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The nonlinear evolution of high-frequency resonant-triad waves in an oscillatory Stokes layer at high Reynolds number

By XUESONG WU

Department of Mathematics, Imperial College, 180 Queens Gate, London SW7 2BZ, UK

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The nonlinear evolution of high-frequency disturbances in high-Reynolds-number Stokes layers is studied. The disturbances are composed of a two-dimensional wave $(2\alpha, 0)$ of magnitude δ, and a pair of oblique waves $(\alpha, \pm\beta)$ of magnitude ϵ, where α, β are the streamwise and spanwise wavenumbers respectively. We assume that $\beta = \surd 3\alpha$ so that the waves form a resonant triad when they are nearly neutral. It is shown that the growth rate of the disturbance is controlled by nonlinear interactions inside 'critical layers'. In order for there to be a nonlinear feedback mechanism between the two-dimensional and the three-dimensional waves, the former is required to have a smaller magnitude than the latter, namely $\delta \sim O(\epsilon^{\frac{4}{3}})$. The timescale of the nonlinear evolution is $O(\epsilon^{-\frac{1}{3}})$.

As in Goldstein & Lee (1992), the amplitude equations turn out to be significantly different from those of Raetz (1959), Craik (1971) and Smith & Stewart (1987) in two respects. Firstly, they are integro-differential equations, i.e. the local growth rate depends on the whole history of the evolution. Secondly the back reaction of the oblique waves on the two-dimensional wave is represented by two cubic terms and one quartic term, rather than by one quadratic term. Our numerical investigations show that the amplitudes of the two- and three-dimensional waves can develop a finite-time singularity, a result of some importance. The structure of the finite-time singularity is identified, and it is found that the two-dimensional wave has a 'more singular' structure than the three-dimensional waves. The finite-time singularity implies that explosive growth is induced by nonlinear effects. We suggest that this nonlinear blow-up of high-frequency disturbances is related to the bursting phenomena observed in oscillatory Stokes layers and can lead to transition to turbulence.

1. Introduction

This study is concerned with the instability of a Stokes layer generated by a sinusoidally oscillating flat plate with speed $U_0 \cos \omega t^*$ in an infinite fluid of kinematic viscosity ν, where t^* is the time. The resulting flow has a boundary-layer thickness $\delta^* = (2\nu/\omega)^{\frac{1}{2}}$, and a Reynolds number $R = U_0(2/\nu\omega)^{\frac{1}{2}}$ based on δ^*. It is an important prototype of unsteady flows as well as an exact solution of the Navier–Stokes equations. Experiments show that the instability of a Stokes layer on a flat wall consists typically of bursts of high-frequency disturbances followed by relaminarization, see e.g. Merkli & Thomann (1975), Hino, Sawamoto & Takasu (1976), Hino *et al.* (1983) and Akhavan, Kamm & Shapiro (1991*a*). This is significantly different from the case of the Stokes layer on a torsionally oscillating cylinder, where the flow can support regular, stationary vortices due to a Taylor–Görtler type of instability (Seminara & Hall 1976).

Since the basic flow is time-periodic, its linearized instability properties are described by equations with time-periodic coefficients, and one might expect Floquet theory to be applicable. Indeed, Floquet theory has successfully predicted the onset of Taylor–Görtler instability. For instance, Hall (1984) and Papageorgiou (1987) have found such instabilities both in the flow generated by a transversely oscillating cylinder, and in oscillatory flow through a curved pipe, respectively. Another successful example of a Floquet analysis is the study of von Kerczek & Davis (1976), where the instability is induced by buoyancy. For the Stokes layer of interest here, the instability is associated with the shear of the basic flow, and a Floquet analysis conducted by Hall (1978) shows that the flow is stable at all the Reynolds numbers investigated, i.e. $R \leqslant 320$; indeed it is possible that according to Floquet theory, the flow is stable at all Reynolds numbers. However, Floquet theory concentrates on the net growth of a disturbance over a whole period, and experimental observations shows that disturbances grow intermittently. This suggests that an approach based on Floquet analysis may not be appropriate if the practical instability of a Stokes layer over a flat plate is to be understood.

An alternative approach is to study the instantaneous instability of the flow. Various heuristic analyses based on a quasi-steady assumption have been suggested, e.g. Collins (1963), Obremski & Morkovin (1969) and Monkewitz (1983). However, as emphasized by Cowley (1987), these methods are mathematically inconsistent in the sense that the Reynolds number is first assumed to be sufficiently large that a multiple-scale analysis can be performed in time, but it is taken to be order one so that viscous effects can be retained in the Orr–Sommerfeld equations. This type of analysis predicts that the instantaneous profile has a lower critical Reynolds number at the start of the acceleration phase than at the end of it (e.g. see von Kerczek & Davis 1974). Akhavan, Kamm & Shapiro (1991*a*, *b*) argued that this was in conflict with the observation that bursts occur at the end of the acceleration phase. They suggested a transition process based on direct numerical simulations and secondary instability arguments. However, they did not address the inception of the modes which are locally the most unstable; moreover there is at least some experimental evidence suggesting that transition seems to be associated with the initial growth of such modes (Merkli & Thomann 1975).

Tromans (1978) and Cowley (1987) put the *ad hoc* quasi-steady assumption on a self-consistent basis by assuming that the Reynolds number was large throughout. They assumed that the most relevant unstable disturbances had frequencies of order $R\omega$, i.e. frequencies much higher than those of the basic flow when $R \gg 1$. The disturbance thus evolves over a very fast timescale $(R\omega)^{-1}$. This is required for mathematical self-consistency, but is also in good agreement with experimental observation. They found that at leading order, the disturbances were governed by the Rayleigh equation. A Rayleigh instability of the Stokes layer to high-frequency disturbances was identified, and the time interval over which such disturbances can grow was calculated.

Wu (1991) investigated nonlinear effects for near-neutral waves in the case of both two-dimensional disturbances, and disturbances consisting of a pair of oblique waves (see also Wu & Cowley 1992; Wu, Lee & Cowley 1992). It was shown that a nonlinear interaction could take place inside the critical layers, i.e. thin regions centred on levels at which the phase velocity of the disturbance is equal to the velocity of the basic flow. This interaction controlled the growth rate, and moreover could induce a finite-time singularity. Consequently, the disturbances could be rapidly amplified by nonlinear effects. This suggests that instability and transition of the Stokes layer may be explained in terms of nonlinear growth over part of a period rather than in terms of the net growth over a whole period. The explosive growth of high-frequency disturbances may be related to the bursts observed in experiments.

A crucial effect that we shall explore here is nonlinearity in the critical layers for the case of a resonant-triad interaction. For a survey of critical-layer analyses, readers are referred to the papers by Stewartson (1981) and Maslowe (1986). These review earlier work both on Rossby waves (e.g. Warn & Warn 1978), and on nonlinear neutral modes of the Benney & Bergeron (1969) type. Here we only review some previous papers which are of immediate relevance to our work.

A pioneering work was that of Hickernell (1984) who considered the evolution of a two-dimensional, free Rossby-wave mode on a shear layer. By imposing an appropriate radiation condition at infinity, the eigenmode is singular because its critical level is not located at the inflexion point of the basic-flow profile. Unlike the forced problem studied by Warn & Warn (1978), the critical-layer dynamics is governed by linear equations at leading order. As a result a weakly nonlinear analysis is possible (cf. Stuart 1960). In particular, by retaining the unsteady term in the leading-order critical-layer equations, Hickernell (1984) obtained an amplitude equation containing a history-dependent nonlinear cubic term. On the other hand, the development of two-dimensional free modes with regular critical layers have been studied by Churilov & Shukhman (1987), Goldstein, Durbin & Leib (1987), Goldstein & Leib (1988) and Goldstein & Hultgren (1989). By developing the unsteady, strongly nonlinear, critical-layer structure of Warn & Warn (1978) they were able to describe the roll-up process of two-dimensional instability waves in shear layers.

Goldstein & Leib (1989) have also considered the viscous evolution of a single three-dimensional oblique mode in a compressible free shear layer. They were concerned with the mode whose critical level coincided with the so-called generalized inflexion point, and hence the eigensolutions for the pressure and normal velocity were regular at the critical layer. They showed that nonlinear effects were associated with a simple pole in the temperature fluctuation, and that this lead to an amplitude evolution equation similar to that of Hickernell (1984) (see also Leib 1991). Their numerical investigations of the amplitude equation reveal that the solution can either blow up at a finite distance downstream or evolve into an equilibrium state. The final state depends on the relative size of the disturbance, the Reynolds number, and on the sign of the real part of the coefficient of the nonlinear term. The same amplitude equation was obtained by Shukhman (1991) in an astrophysical instability problem.

The evolution of a pair of interactive oblique modes with the same streamwise wavenumber, but spanwise wavenumbers of opposite sign, was studied by Goldstein & Choi (1989) for a free shear layer. Again, weakly nonlinear critical-layer effects of Hickernell (1984) type were found; this is in contrast to the strongly nonlinear critical layer in the corresponding two-dimensional case (e.g. Goldstein & Leib 1988). Primarily this is because the streamwise velocity has a simple pole type of singularity. This case is also significantly different from the case of a single oblique wave (Goldstein & Leib 1989), because in the latter case the nonlinear effects associated with the pole singularity can be suppressed by an appropriate Squire transformation. Recently, Wu *et al.* (1992) have extended the analysis of Goldstein & Choi (1989) by including viscosity.

In this paper, we shall be concerned with a 'resonant-triad' interaction. This is a 'stronger' interaction in the sense that quadratic interactions can affect the development of the disturbance (cf. the cubic nonlinearity in Hickernell's 1984 weakly nonlinear analysis). This type of interaction takes place in many kinds of flows, e.g. see Craik (1985). The possibility of resonant-triad interactions between Tollmien–Schlichting (T-S) waves was suggested by Raetz (1959) and Craik (1971). They proposed the following mechanism. Suppose there is a pair of three-dimensional modes, say $(\alpha, \pm\beta, c_1)$, and a two-dimensional mode, say $(2\alpha, 0, c_2)$, where $\pm\beta$ are the spanwise wavenumbers of the three-dimensional modes, α and 2α are the streamwise wavenumbers of the three-dimensional and two-dimensional modes respectively, and c_1 and c_2 are the respective phase velocities. If

$$c_1 = c_2, \tag{1.1}$$

then a nonlinear interaction can occur at order (amplitude)^2, i.e. 'sooner' than the order-(amplitude)^3 interaction of the non-resonant case (Stuart 1960; Watson 1960). If (1.1) holds only approximately, then the waves are said to be detuned. In this paper, we shall restrict our attention to the tuned resonant case. Goldstein & Lee (1992) allow for a detuned triad, although we note that a more general approach would be to allow for modulations in the spanwise direction (cf. Craik 1985). Similarly the analysis could be generalized by studying a wavepacket localized in space, i.e. by allowing for modulations in the streamwise direction.

We also recall that resonant-triad modes may have more general forms and do not necessarily have to be associated with the same instability (e.g. Craik 1985). For instance, Hall & Smith (1988) considered an interesting resonant-triad interaction in concave channel flow. This flow can support both T-S waves and Taylor–Görtler vortices. The triad they considered was composed of a pair of oblique T-S waves and a Görtler vortex. Recently Thomas (1992) has studied resonant triads composed of T-S waves and various 'wall modes' for flows over compliant walls.

As pointed out by Stuart (1962*a*, *b*), condition (1.1) is not readily satisfied by T-S waves at finite Reynolds numbers. Craik (1971) found that in a boundary layer, for a given α, (1.1) is valid only for certain specific values of β, and he referred to this as the selective amplification mechanism. However, at asymptotically high Reynolds numbers Stuart's argument seems to have less force. Smith & Stewart (1987) put Craik's analysis on an asymptotic basis by using a triple-deck style of analysis; they showed that the resonant-triad interaction took place for 'high-frequency' T-S waves, i.e. for high-frequency solutions to the triple-deck equations.

In the case of Rayleigh waves, condition (1.1) can be easily satisfied. Suppose $(2\alpha, 0, c)$ is a neutral mode of Rayleigh's equation, then from Squire's transformation, it follows that if

$$\beta = \surd 3\alpha, \tag{1.2}$$

the $(\alpha, \pm\beta, c)$ modes are also neutral, and moreover the resonant-triad condition is satisfied. The relation (1.2) was given by Craik (1971), and is also the resonant condition for high-frequency T-S waves at asymptotically high Reynolds numbers (e.g. see Smith & Stewart 1987).

Recently, Goldstein & Lee (1992) studied a resonant-triad interaction in a boundary layer. They assumed that a weak adverse pressure gradient was present, so that long-wavelength Rayleigh waves existed (see also Goldstein *et al.* 1987). As in Hickernell (1984), the critical layers were of unsteady/non-equilibrium type. It was shown that in order for there to be a weakly nonlinear mutual interaction between the two-dimensional and three-dimensional waves, the amplitudes of the three-dimensional waves were required to be much larger than that of the two-dimensional wave. Moreover, they pointed out that the feedback effect of the three-dimensional waves on the two-dimensional wave was through a quartic rather than a quadratic interaction; in particular the amplitude equation for the two-dimensional wave contained a quartic term. The solutions of the amplitude equations were found to develop a singularity at a finite distance downstream.

Goldstein & Lee (1992) also showed that if the two-dimensional and three-dimensional waves were of equal size, then the growth rate of the three-dimensional waves was enhanced by the two-dimensional wave, but the three-dimensional waves had *no* back reaction on the two-dimensional wave. As a result, the three-dimensional waves exhibited faster-than-exponential growth, while the two-dimensional wave continued to grow exponentially. Eventually, the three-dimensional waves become sufficiently large to affect the two-dimensional wave, at which point the evolution entered the fully interactive resonant-triad stage. We note that the super-exponential growth can in fact occur even when the three-dimensional waves are very much smaller than the two-dimensional waves. This therefore provides a selective amplification mechanism for three-dimensional waves.

A fully interactive resonant-triad of T-S waves in the Blasius boundary layer was proposed by Mankbadi (1992). He was concerned with the upper-branch scaling regime (cf. Smith & Stewart 1987), which covers almost the entire range of unstable Reynolds number. In this regime, the viscous critical layer is distinct from the viscous wall layer and hence the flow is not described by a triple-deck structure (e.g. Bodonyi & Smith 1981). He showed that the dominant nonlinearity occurs in the critical layer. The importance of critical layers in resonant-triad interactions was shown earlier by Usher & Craik (1975) in a more heuristic way.

The present study was originally inspired by an early version† of Goldstein & Lee (1992). Our aim is to examine the fully interactive resonant-triad instability of a Stokes layer, and its relevance to transition – with special reference to its bursting nature. For convenience, we assume that the Reynolds number is sufficiently large so that the critical layers involved in our analysis are predominantly *inviscid* and *unsteady*, as in Goldstein & Lee (1992). This is in contrast to the steady viscous critical layer of Mankbadi (1992).‡ The nonlinear terms in the amplitude equations, as will be shown, are determined by nonlinear interactions inside the critical layers and do not depend on the basic flow as a whole. Thus the results obtained are valid for Rayleigh instability in a wide class of shear flows. Indeed, we could formulate our analysis for general flows; however, we believe that the example of the Stokes layer is sufficiently illustrative. Finally we note that the Rayleigh instability waves in Goldstein & Lee (1992) have long wavelengths, while in our problem, the Rayleigh modes are assumed to have order-one wavelengths. Despite this difference the amplitude equations turn out to be essentially the same as those of Goldstein & Lee (1992); the only differences are in the coefficients. We shall discuss this aspect in §5.

The paper is organized as follows. In §2 the underlying scaling is derived and the problem is formulated. Specifically, we show how nonlinear effects inside the critical layers come into play, and how the disturbances gradually evolve into a nonlinear stage from the strictly linear finite-growth-rate stage. In §3 the outer expansions are

† In this earlier version, important features of the fully interactive interactions were highlighted, but not fully explored by detailed asymptotic expansions.

‡ However, we note that Mankbadi (1992) may have omitted an unsteady diffusion layer, which surrounds the viscous critical layer. The present author is investigating this issue with Dr Mankbadi.

carried out. The asymptotic solutions of the outer problem near the critical layers are obtained, and as usual they contain some undetermined 'jumps'. The main results of this section are two solvability conditions which have to be imposed on higher-order inhomogeneous Rayleigh equations. In §4 the inner expansions within the critical layers are carried out and the solutions, together with their asymptotes, are obtained. The jumps are obtained by matching with the outer solutions. In §5, using the solvability conditions and the calculated jumps, we determine two coupled integro-differential equations for the disturbance amplitudes. Expressions for the coefficients in these equations are given, and they are evaluated for the specific example of a Stokes layer. Numerical solutions of the amplitude equations suggest that a finite-time singularity can form. The structure of this singularity is identified and discussed. Finally, in §6 we draw some conclusions and discuss implications of the present study. Since the analysis is rather lengthy, readers may find it helpful to look at the main result of the analysis, i.e. the amplitude equations (5.1) and (5.5), before studying the details given in §§3 and 4. The main task of the analysis is to derive the nonlinear kernels and the associated coefficients of the nonlinear terms.

2. Scaling arguments and formulation

The flow is described by non-dimensional Cartesian coordinates (x, y, z), where x is parallel to the direction of oscillation of the plate, y is normal to the plate, and z is the spanwise direction. If we take δ^*, ω^{-1}, and U_0 as characteristic length, time and velocity scales respectively, then the basic flow is given by

$$(\bar{U}, \bar{V}, \bar{W}) = (\cos(\tau - y)\,\mathrm{e}^{-y}, 0, 0),$$

where $\tau = \omega t^*$, with t^* being the dimensional time variable. We denote the perturbed flow by

$$(\bar{U} + u, v, w).$$

Following Tromans (1978) and Cowley (1987), we study the high-frequency instability waves, and introduce the fast timescale

$$t = R\tau \tag{2.1}$$

to take account of the frequency of Rayleigh waves. At any time, there exist an infinite number of Rayleigh modes (Cowley 1987), but the most rapidly growing modes can for the most part be found at those times and for those wavenumbers that lie beneath the solid curve A in figure 1. In this paper, we shall concentrate on these modes.

We assume that the velocity component v_{2D} of the two-dimensional disturbance has a magnitude of $O(\delta)$, and that to the leading order has the form

$$v_{2D} \sim \delta B_0 \phi_2(y, \tau) E^2 + \text{c.c.},$$

where $\delta \ll 1$, and the order-one complex constant B_0 is a measure of the scaled amplitude of the two-dimensional wave. Throughout the paper c.c. represents the complex conjugate of the part written out explicitly. For convenience, we have defined

$$E = \exp(\mathrm{i}\alpha x - \mathrm{i}\hat{\theta}(t)), \tag{2.2}$$

where

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = \tfrac{1}{2}\alpha c(\tau) + \tfrac{1}{2}R^{-\frac{1}{2}}\Omega_1(\tau) + \ldots, \tag{2.3}$$

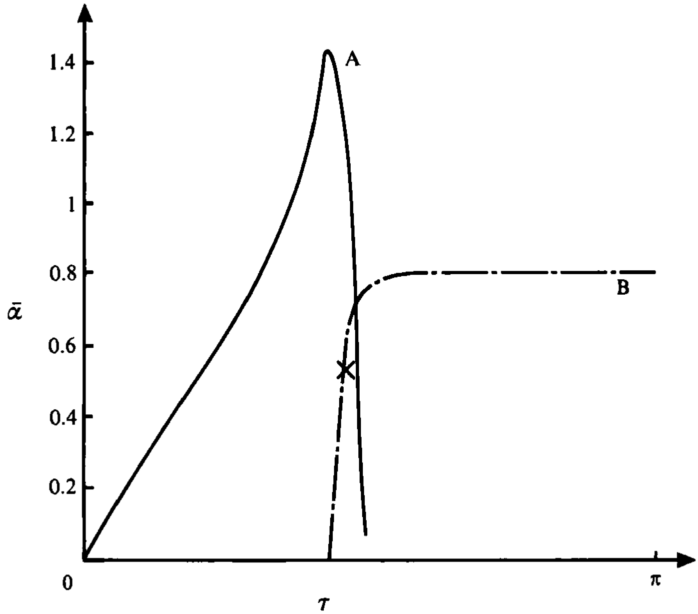

FIGURE 1. Sketch of the neutral diagram for wavenumbers $\bar{\alpha} = 2\alpha$ (from Cowley 1987). τ_0 is a point on the right-hand branch of the neutral curve A. In the analysis we concentrate on times close to $\tau = \tau_0 + \epsilon^{\frac{1}{3}}\tau_1$, where $\tau_1 < 0$ and ϵ is the magnitude of the oblique waves. The point marked by $\times$ is a mode crossing point; see Cowley (1987) for details.

with c being a complex speed, i.e. $c = c_r + ic_i$. In this study, we assume that the wavenumber α is of order one.

The three-dimensional disturbance is composed of a pair of oblique Rayleigh waves with equal amplitudes ϵ and equal streamwise wavenumbers α, but opposite spanwise wavenumbers $\pm\beta$. We assume that $\beta = \sqrt{\alpha}$ so that the oblique waves have the same phase velocity as the two-dimensional wave, i.e. we satisfy the resonant condition. The velocity component v_{3D} of the three-dimensional disturbance can be expressed, to leading order, in the form

$$v_{3D} \sim \epsilon A_0 v_1(y, \tau)\, E \cos\beta z + \text{c.c.},$$

where A_0 is a measure of the scaled amplitude of the three-dimensional waves (for simplicity we assume that the two oblique waves have equal amplitudes, although this restriction could be relaxed).

According to linear theory both the two- and three-dimensional disturbances can grow over the same part of the Stokes-layer oscillation period, say from a time τ_i until a time τ_0, at which the waves become neutral (see figure 1). In the vicinity of the neutral time τ_0, the linear growth rates are small, and critical layers exist. Owing to the singular nature of Rayleigh's equation near such critical layers, nonlinearity first becomes important there (see references cited in §1).

Before proceeding to a formal asymptotic analysis, we must obtain the underlying scaling, i.e. the relationships between c_i, ϵ and δ. For this purpose, we introduce the intermediate timescale $t_1 = \alpha c_i t$ which represents the 'slow' timescale over which the wave grows. This timescale is much 'slower' than the 'high-frequency' timescale of the almost neutral instability waves, but much 'faster' than the slow timescale over which the underlying Stokes flow oscillates. The disturbances are then described by the timescales t, t_1, and τ. (In fact there is a fourth timescale in the problem, namely $\alpha c_i R^{-1}$, over which the growth rate changes (see (2.8) and (3.7)). However, for the purpose of our scaling arguments we do not need to consider this timescale.) In terms of these new variables, the x-momentum equation can be written as follows:

$$\left\{\alpha c_i \frac{\partial}{\partial t_1} + (\bar{U}-c_r)\frac{\partial}{\partial x} + 2R^{-1}\frac{\partial}{\partial \tau}\right\} u + v\frac{\partial \bar{U}}{\partial y} = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}uu - \frac{\partial}{\partial y}uv - \frac{\partial}{\partial z}uw,$$

where $\partial/\partial t$ has been replaced by $-c_r\,\partial/\partial x$ since this is true to the order required (see (2.2) and (2.3)).

We are interested in the case where the nonlinear evolution occurs on the fastest possible timescale, i.e. we require that the time-variation term appears at leading order in the critical-layer equations (cf. Hickernell 1984). Suppose that the thickness of the critical layers is $O(\mu)$, then near a critical layer $y = y_c$, where $\bar{U}(y_c) = c_r$, we have that $(\bar{U}-c_r) = O(\mu)$, if we assume that $\bar{U}_y(y_c) \neq 0$. By balancing the $\alpha c_i\,\partial u/\partial t_1$ and $(\bar{U}-c_r)\,\partial u/\partial x$ terms, we conclude that

$$\mu \sim O(c_i). \tag{2.4}$$

As in Goldstein & Lee (1992), this scaling brings in an unsteady/non-equilibrium effect that does not occur in the analyses of Raetz (1959), Craik (1971), Smith & Stewart (1987) and Mankbadi (1992). As pointed out by Hickernell (1984), such a non-equilibrium term leads to an amplitude equation that involves history effects and takes the form of an integro-differential equation.

As a critical layer is approached, the streamwise velocity of the three-dimensional waves exhibits a pole type of singularity, i.e.

$$u_{3D} \sim \epsilon/(y-y_c),$$

as found by Benney (1961), and as was implied by Squire's (1933) analysis (see also Goldstein & Choi 1989). Hence in the critical layer, the magnitude of $u_{3D,y} = O(\epsilon\mu^{-2})$. Moreover, according to the asymptotic properties of solutions to Rayleigh's equation, the normal velocities of the two- and three-dimensional waves are of order δ and ϵ respectively, both in the critical layer and in the main part of the flow. The nonlinear interaction inside the critical layer between the two-dimensional and the three-dimensional waves through the $(\partial/\partial y)\ (v_{2D}u_{3D})$ term (the forcing from the

$$(\partial/\partial x)\ (u_{2D}u_{3D}),\ (\partial/\partial y)\,(u_{2D}v_{3D}),$$

etc. terms is much smaller) produces a forcing term of $O(\epsilon\delta\mu^{-2})$ in the x-momentum equation. By balancing this forcing with the $\alpha c_i\,\partial y_{3D}/\partial t_1$ term, we conclude that it drives a three-dimensional wave component with a streamwise velocity of $O(\epsilon\delta\mu^{-3})$. In order for the evolution of the three-dimensional waves to be affected by a quadratic resonant interaction, we require that this nonlinearly generated velocity smooths out the $O(\epsilon\alpha c_i)$ discontinuity in the outer solution, i.e. $\epsilon\delta\mu^{-3} \sim \epsilon\alpha c_i$ (cf. Hickernell, 1984; Goldstein & Lee, 1992). This balance yields

$$\mu \sim O(\delta^{\frac{1}{4}}). \tag{2.5}$$

We shall concentrate on the situation where there is energy feedback between the two-dimensional and three-dimensional waves. Conventionally, the quadratic interaction of the oblique waves affects the growth of the two-dimensional wave. In the present case this interaction takes place inside the critical layers, and produces a forcing of $O(\epsilon^2\mu^{-2})$ through the $(\partial/\partial x)(u_{3D}^2)$, $(\partial/\partial y)\ (u_{3D}v_{3D})$, and $(\partial/\partial z)\ (u_{3D}w_{3D})$

terms. By balancing these terms with the $\alpha c_i\, \partial u_{2D}/\partial t_1$ term, we find that it produces both a two-dimensional wave component and a mean-flow distortion, say u_M, of $O(\epsilon^2\mu^{-3})$. However, the two-dimensional wave component so generated does not produce a velocity jump across the critical layer. As a result, the quadratic interaction between the three-dimensional waves does not affect the growth of the two-dimensional fundamental wave (see Goldstein & Lee 1992 and also §4 below). The cubic interaction between the two-dimensional wave and the mean-flow distortion through the $(\partial/\partial y)(v_{2D}\, u_M)$ term, generates a forcing of $O(\epsilon^2\delta\mu^{-4})$. Balancing this forcing with the $\alpha c_i\, \partial u_{2D}/\partial t_1$ term, we find that it drives an $O(\epsilon^2\delta\mu^{-5})$ two-dimensional wave component. We find that this two-dimensional wave component contributes a velocity jump of the same order (see §4). Moreover, the cubic interaction will affect the two-dimensional wave when this jump matches the $O(\delta\alpha c_i) \sim O(\delta\mu)$ discontinuity of the outer solution for the two-dimensional wave. Thus we require $\epsilon^2\delta\mu^{-5} \sim \delta\mu$, i.e.

$$\mu \sim O(\epsilon^{\frac{1}{3}}). \tag{2.6}$$

Hence from (2.5) we have that

$$\delta \sim O(\epsilon^{\frac{4}{3}}). \tag{2.7}$$

Equations (2.6) and (2.7) fix the underlying scaling for our analysis. Note that the magnitude of the three-dimensional waves is required to exceed that of the two-dimensional wave. This is as found by Goldstein & Lee (1992) and Mankbadi (1992), but it is in contrast to much previous work on three-dimensional instability where it is generally assumed that the magnitude of three-dimensional waves does not exceed that of two-dimensional wave. However, a point in favour of the present scaling is the fact that for Stokes layers it has not yet been possible to identify a dominant two-dimensional instability stage by significantly reducing the background disturbance and introducing an artificial controlled two-dimensional disturbance.

With the scaling fixed as above, it follows that we should concentrate on times close to

$$\tau = \tau_0 + \epsilon^{\frac{1}{3}}\tau_1, \tag{2.8}$$

namely the time at which the linear growth rate, according to a Taylor expansion, $\alpha c_\tau = \alpha c(\tau_0) + \epsilon^{\frac{1}{3}}\alpha c_\tau(\tau_0)\tau_1 + \ldots$, is reduced to $O(\epsilon^{\frac{1}{3}})$. It is at this stage that nonlinearity begins to affect the evolution of our flow. We note that the disturbance evolves over timescale t_1, while the linear growth rate and the basic flow depend parametrically on the slow, and very slow, timescales τ_1 and τ respectively. Since the basic flow $\bar{U}$ changes only on the very slow timescale τ, it is sufficient to expand its profile at τ in a Taylor series about τ_0:

$$\bar{U}(y,\tau) = \bar{U}(y,\tau_0) + \epsilon^{\frac{1}{3}}U_\tau(y,\tau_0)\,\tau_1 + \ldots.$$

Hereafter, all basic flow quantities will be evaluated at τ_0 unless otherwise mentioned. For convenience, we rewrite the nonlinear evolution timescale t_1 in the form

$$t_1 = \tfrac{1}{2}\epsilon^{\frac{1}{3}}t. \tag{2.9}$$

The evolution picture is summarized in figure 2 for the case when the evolution becomes nonlinear near the right-hand branch of curve A figure 1 (our analysis is also valid near the left-hand branch). As illustrated, we assume that the disturbances are initially described by linear theory and grow exponentially. Then as the neutral time τ_0 is approached, the growth rates become small, and linear critical layers emerge. As

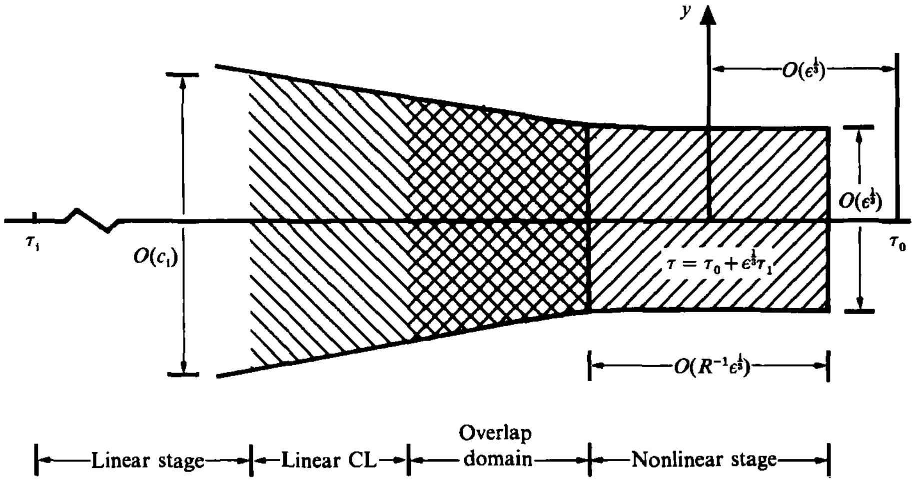

FIGURE 2. Evolution stages and critical-layer structures. The disturbances initially grow exponentially according to linear theory. As they approach the neutral time τ_0, the growth rates become small, and linear critical layers emerge. As the growth rates decrease further to $O(\epsilon^{\frac{1}{3}})$, nonlinear interactions inside the critical layers control the overall development of the disturbance. The earlier linear and the subsequent nonlinear evolution stages are required to match in the overlapping domain.

the growth rate decreases further to $O(\epsilon^{\frac{1}{3}})$, i.e. at times $\tau_1 = O(1)$, nonlinear interactions take place and control the overall development of the disturbance. The earlier linear and the subsequent nonlinear solutions are required to match in a mutual domain of overlap.

We close this section by listing the assumptions of our study.

(i) The Reynolds number is assumed to be large throughout so that the disturbances evolve on a much faster timescale than that of the basic flow. Such an assumption is crucial for self-consistency of the quasi-steady theory.

(ii) The disturbances are assumed to have sufficiently small amplitudes so that linear theory provides a valid description initially. Nonlinearity asserts itself when the linear growth rates become small. This means that we shall effectively consider nonlinear effects on slowly modulated neutral modes. One may well have reason to argue that the nonlinear evolution of disturbances with an order-one growth rate would be more realistic. However, this can only occur when the disturbances have an order-one magnitude as well; a fully nonlinear theory then seems necessary. Indeed if it is assumed that the disturbances have a small magnitude but order-one growth rate, one then meets an inherent difficulty in pursuing a self-consistent treatment (e.g. see Craik 1985, p. 193). Thus for consistency, almost all nonlinear theories have to concentrate on nearly neutral modes of small amplitude. Nevertheless, in the present study the nonlinear evolution still occurs over a much faster timescale than that of the basic flow.

(iii) The linear growth rate is assumed to be a leading-order effect in the critical layers. The singularity at the critical layers is then removed by this unsteady effect. For simplicity, we assume that the Reynolds number is large enough, i.e.

$$R \gg \epsilon^{-1}$$

so that viscosity does not play a leading role in the critical layers.

(iv) In order for there to be an energy feedback mechanism between the planar and oblique waves, their magnitudes are assumed to satisfy relation (2.7). For simplicity we also assume that the oblique modes have equal amplitude, although an obvious, if algebraically messy, extension would be to relax this restriction.

(v) We assume that the disturbance is modulated neither in the spanwise nor in the streamwise direction. An obvious extension would be to examine such wavepackets.

Despite these restrictions, we believe that our theoretical analysis yields at least a partial explanation of the transition features of Stokes layers and other shear layers.

3. Outer expansions

3.1. *Outer expansions and solutions*

Outside the critical layers, the unsteady flow is basically linear and is governed, to the required order of approximation, i.e. $O(\epsilon^{\frac{5}{3}})$, by the following equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{3.1}$$

$$2R^{-1}\frac{\partial u}{\partial \tau}+\bar{U}\frac{\partial u}{\partial x}+v\frac{\partial \bar{U}}{\partial y} = -\frac{\partial p}{\partial x}, \tag{3.2}$$

$$2R^{-1}\frac{\partial v}{\partial \tau}+\bar{U}\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}, \tag{3.3}$$

$$2R^{-1}\frac{\partial w}{\partial \tau}+\bar{U}\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z}. \tag{3.4}$$

By eliminating the pressure terms, we obtain

$$\left(2R^{-1}\frac{\partial}{\partial \tau}+\bar{U}\frac{\partial}{\partial x}\right)\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)-\frac{\partial \bar{U}}{\partial y}\frac{\partial w}{\partial x} = 0, \tag{3.5}$$

and

$$\left(2R^{-1}\frac{\partial}{\partial \tau}+\bar{U}\frac{\partial}{\partial x}\right)\nabla^2 v-\bar{U}_{yy}\frac{\partial v}{\partial x} = 0. \tag{3.6}$$

With the multiple timescales introduced in §2, the time derivative $\partial/\partial\tau$ should be interpreted as follows

$$\frac{\partial}{\partial \tau}\to R\frac{\partial}{\partial t}+\tfrac{1}{2}R\epsilon^{\frac{1}{3}}\frac{\partial}{\partial t_1}+\epsilon^{-\frac{1}{3}}\frac{\partial}{\partial \tau_1}+\frac{\partial}{\partial \tau_0}. \tag{3.7}$$

The velocity (u, v, w) and the pressure p of the disturbances are expanded as follows:

$$u = \epsilon u_1+\epsilon^{\frac{4}{3}}u_2+\epsilon^{\frac{5}{3}}u_3+\dots, \tag{3.8}$$

$$v = \epsilon v_1+\epsilon^{\frac{4}{3}}v_2+\epsilon^{\frac{5}{3}}v_3+\dots, \tag{3.9}$$

$$w = \epsilon w_1+\epsilon^{\frac{4}{3}}w_2+\epsilon^{\frac{5}{3}}w_3+\dots, \tag{3.10}$$

$$p = \epsilon p_1+\epsilon^{\frac{4}{3}}p_2+\epsilon^{\frac{5}{3}}p_3+\dots. \tag{3.11}$$

The presence of the two-dimensional disturbance requires that the expansions must be taken to one order higher than is necessary for a pair of oblique waves (Wu 1991;

Goldstein & Choi 1989; Wu *et al.* 1992). However, we shall see that the analysis at the first two orders are the same.

The earlier linear solution suggests that v_1 can be written as

$$v_1 = A(t_1, \tau_1, \tau_0)\, \bar{v}_1(y, \tau_0) \cos\beta z E + \text{c.c.}, \tag{3.12}$$

where $A(t_1)$ is the amplitude of the three-dimensional waves, and E is defined by (2.2) and (2.3), provided that τ in (2.3) is replaced by τ_0 (recall that the analysis is only valid in the vicinity of a 'neutral' time). Here the dependence on τ_0 is due to the time variation of the basic flow, while the dependence of A on τ_1 is due to the time variation of the linear growth rate. Because both τ_0 and τ_1 appear only as parameters to the order that we work, they will not be written out explicitly hereafter.

The function $\bar{v}_1$ satisfies Rayleigh's equation:

$$(\bar{U}-c)(\mathrm{D}^2-\bar{\alpha}^2)\bar{v}_1 - \bar{U}_{yy}\bar{v}_1 = 0, \tag{3.13}$$

where we have written

$$\bar{\alpha} = (\alpha^2+\beta^2)^{\frac{1}{2}}.$$

The boundary conditions are that $\bar{v}_1$ vanishes both at wall and at infinity. For definiteness we will concentrate on times τ_0 on the right-hand side of curve A. In this case there exist two critical levels, neither of which is at an inflexion point. The jump in Reynolds stress across each critical layer is non-zero, although the sum of the jumps is zero.

Let $\eta = y - y_c^j$, where y_c^j is the jth critical level; then as $\eta \to \pm 0$, $\bar{v}_1$ has the following asymptotic solution:

$$\bar{v}_1 \sim a_j^{\pm}\, \phi_a + b_j^{\pm}[\phi_b + p_j\, \phi_a \log|\eta|], \tag{3.14}$$

where $\phi_a = \eta + \frac{1}{2}p_j\,\eta^2 + \ldots$ and $\phi_b = 1 + q_j\,\eta^2 + \ldots$.

The function v_2 has the form

$$v_2 = \bar{v}_2(y, t_1)\cos\beta z E + B(t_1)\,\phi_2(y)\,E^2 + v^{(0,2)}(y, t_1)\cos\beta z + \text{c.c.} + \ldots, \tag{3.15}$$

where $B\phi_2 E^2$ is a two-dimensional wave with scaled amplitude $B(t_1)$, and $\bar{v}_2$ is the deviation of the eigenfunction of the three-dimensional waves from the neutral state. The function ϕ_2 satisfies Rayleigh's equation, and we have assumed that $2\alpha = \bar{\alpha}$; thus

$$\phi_2 = \bar{v}_1. \tag{3.16}$$

As for a pair of oblique waves (Wu 1991), $\bar{v}_2$ satisfies the inhomogeneous Rayleigh equation:

$$\left[\mathrm{D}^2 - \left(\bar{\alpha}^2 + \frac{\bar{U}_{yy}}{\bar{U}-c}\right)\right]\bar{v}_2 = (\mathrm{i}\alpha)^{-1}\left\{\left[-\frac{\mathrm{d}A}{\mathrm{d}t_1} - (\mathrm{i}\alpha\bar{U}_\tau\tau_1)A\right]\frac{\bar{U}_{yy}}{(\bar{U}-c)^2} + \frac{\mathrm{i}\alpha\bar{U}_{yy\tau}\tau_1 A}{\bar{U}-c}\right\}\bar{v}_1. \tag{3.17}$$

The asymptotic form of $\bar{v}_2$ is

$$\bar{v}_2 \sim -b_j^{\pm}r_j\log|\eta| + (a_j^{\pm}\, r_j + b_j^{\pm}\, s_j)\,\eta\log|\eta| + \ldots + c_j^{\pm}\,\phi_a + d_j^{\pm}[\phi_b + p_j\,\phi_a\log|\eta|], \tag{3.18}$$

where
$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y}, \quad q_j = \tfrac{1}{2}\bar{\alpha}^2 + \frac{1}{2}\frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}^2_{yy}}{\bar{U}^2_y}, \tag{3.19}$$

$$r_j = (\mathrm{i}\alpha)^{-1}\frac{\bar{U}_{yy}}{\bar{U}^2_y}\left[-\frac{\mathrm{d}A}{\mathrm{d}t_1} - (\mathrm{i}\alpha\bar{U}_\tau\tau_1)A\right], \tag{3.20}$$

$$s_j = (\mathrm{i}\alpha)^{-1}\left\{(-\mathrm{i}\alpha\bar{U}_{y\tau}\tau_1 A)\frac{\bar{U}_{yy}}{\bar{U}^2_y} + (\mathrm{i}\alpha\tau_1 A)\frac{\bar{U}_{yy\tau}}{\bar{U}_y} + \left(-\frac{\mathrm{d}A}{\mathrm{d}t_1} - \mathrm{i}\alpha\bar{U}_\tau\tau_1 A\right)\frac{\bar{U}_y\bar{U}_{yyy} - \bar{U}^2_{yy}}{\bar{U}^3_y}\right\}. \tag{3.21}$$

All basic flow quantities are evaluated at time τ_0 and at the critical level y^j_c.

At $O(\epsilon^{\frac{5}{3}})$, it is sufficient to solve for the deviation of the two-dimensional wave eigenfunction from its neutral state; for this purpose we write

$$\bar{v}_3 = \phi_3(y, t_1)E^2 + \text{c.c.} + \dots. \tag{3.22}$$

The function ϕ_3 satisfies an inhomogeneous Rayleigh equation similar to that for $\bar{v}_2$, namely

$$\left[\mathrm{D}^2 - \left(\bar{\alpha}^2 + \frac{\bar{U}_{yy}}{\bar{U}-c}\right)\right]\phi_3 = (2\mathrm{i}\alpha)^{-1}\left\{\left[-\frac{\mathrm{d}B}{\mathrm{d}t_1} - (2\mathrm{i}\alpha\bar{U}_\tau\tau_1)B\right]\frac{\bar{U}_{yy}}{(\bar{U}-c)^2} + \frac{2\mathrm{i}\alpha\bar{U}_{yy\tau}\tau_1 B}{\bar{U}-c}\right\}\phi_2. \tag{3.23}$$

The asymptotic behaviour of ϕ_3 as $\eta \to \pm 0$ is

$$\phi_3 \sim -b^{\pm}_j R_j \log|\eta| + (a^{\pm}_j R_j + b^{\pm}_j S_j)\,\eta\log|\eta| + \dots + C^{\pm}_j\phi_a + D^{\pm}_j[\phi_b + p_j\phi_a\log|\eta|], \tag{3.24}$$

where
$$R_j = (2\mathrm{i}\alpha)^{-1}\frac{\bar{U}_{yy}}{\bar{U}^2_y}\left[-\frac{\mathrm{d}B}{\mathrm{d}t_1} - (2\mathrm{i}\alpha\bar{U}_\tau\tau_1)B\right], \tag{3.25}$$

$$S_j = (2\mathrm{i}\alpha)^{-1}\left\{(-2\mathrm{i}\alpha\bar{U}_{y\tau}\tau_1 B)\frac{\bar{U}_{yy}}{\bar{U}^2_y} + (2\mathrm{i}\alpha\tau_1 B)\frac{\bar{U}_{yy\tau}}{\bar{U}_y} + \left(-\frac{\mathrm{d}B}{\mathrm{d}t_1} - 2\mathrm{i}\alpha\bar{U}_\tau\tau_1 B\right)\frac{\bar{U}_y\bar{U}_{yyy} - \bar{U}^2_{yy}}{\bar{U}^3_y}\right\}. \tag{3.26}$$

Note that $C^{\pm}_j$ and $D^{\pm}_j$ are generally different from $c^{\pm}_j$ and $d^{\pm}_j$ respectively, since the right-hand side of (3.17) differs from that of (3.23). The jumps $(a^+_j - a^-_j)$ etc. will be determined by analysing the critical layers.

According to the continuity equation, we can write $w_1 = A(t_1)\bar{w}_1(y)E\sin\beta z + \text{c.c.}$; then $\bar{w}_1$ satisfies

$$\frac{\partial\bar{w}_1}{\partial y} + \frac{\bar{U}_y}{\bar{U}-c}\bar{w}_1 = -\beta\bar{v}_1,$$

the solution of which is $\bar{w}_1 = \bar{\alpha}^{-1}\sin\theta[\bar{U}_y/(\bar{U}-c)\,\bar{v}_1 - \bar{v}_{1,y}]$, (3.27)

where $\sin\theta = \beta/\bar{\alpha} = \tfrac{1}{2}\surd 3.$

The velocity u_1 takes the form

$$u_1 = A(t_1)\,\bar{u}_1(y)\,E\cos\beta z + \bar{u}^{(0,2)}_1(y, t_1)\cos 2\beta z + \text{c.c.},$$

where $\bar{u}_1$ can be obtained from the continuity equation:

$$\bar{u}_1 = -(\mathrm{i}\alpha)^{-1}\{[\bar{U}_y/(\bar{U}-c)\,\bar{v}_1 - \bar{v}_{1,y}]\sin^2\theta + \bar{v}_{1,y}\}. \tag{3.28}$$

Note that as in Goldstein & Choi (1989), a spanwise-dependent mean-flow component

$\bar{u}_1^{(0,2)}(y, t_1)\cos 2\beta z$ has to be included in order to match with the inner solution. As will be shown later, this is driven by the slip velocity generated by nonlinear interactions inside the critical layers. In turn it drives a longitudinal vortex in the main part of the flow.

Similarly $p_1 = A(t_1)\,\bar{p}_1 E\cos\beta z + \text{c.c.}$, and

$$\bar{p}_1 = \mathrm{i}\bar{\alpha}^{-1}\cos\theta[\bar{U}_y \bar{v}_1 - (\bar{U}-c)\,\bar{v}_{1,y}]. \tag{3.29}$$

As $y \to y_c^j$, the asymptotic behaviour of $\bar{p}_1, \bar{u}_1, \bar{w}_1$ can be shown to be

$$\bar{p}_1 \sim \mathrm{i}\bar{\alpha}^{-1}\bar{U}_y\cos\theta b_j^{\pm} + \ldots, \quad \bar{u}_1 \sim -(\mathrm{i}\alpha)^{-1}\sin^2\theta b_j^{\pm}\eta^{-1} + \ldots, \quad \bar{w}_1 \sim \bar{\alpha}^{-1}\sin\theta b_j^{\pm}\eta^{-1} + \ldots.$$

It is worth noting that the singularity in $\bar{u}_1$ is a simple pole, rather than the logarithmic branch point which is characteristic of the two-dimensional case. It is this difference that results in the faster nonlinear evolution timescale in the three-dimensional case. For free shear layers, Goldstein & Choi (1989) have shown that because of this pole type of singularity, the critical-layer dynamics of two-dimensional and three-dimensional disturbances are significantly different. In the former case the dynamics is strongly nonlinear (e.g. see Goldstein & Leib 1988), while in the latter case it is weakly nonlinear. Craik (1971) pointed out that this pole was the reason why the quadratic interaction coefficients for the oblique waves were $O(R)$.

At the next order in the pressure perturbation, we only need to solve for the two-dimensional wave component; therefore we write

$$p_2 = B(t_1)\,\bar{p}_2(y)\,E^2 + \text{c.c.} + \ldots. \tag{3.30}$$

It is found that

$$\bar{p}_2 = \tfrac{1}{2}\mathrm{i}\alpha^{-1}[\bar{U}_y\phi_2 - (\bar{U}-c)\phi_{2,y}], \tag{3.31}$$

and that as $y \to y_c^j$,

$$\bar{p}_2 \sim \tfrac{1}{2}\mathrm{i}\alpha^{-1}\bar{U}_y b_j^{\pm} + \ldots. \tag{3.32}$$

We now introduce an inner variable:

$$Y = \eta/\epsilon^{\frac{1}{3}}. \tag{3.33}$$

The outer expansions written in terms of the inner variable are as follows:

$$\begin{aligned} v \sim{}& \epsilon b_j^{\pm}AE\cos\beta z + \epsilon^{\frac{4}{3}}\log\epsilon^{\frac{1}{3}}(-b_j^{\pm}r_j + b_j^{\pm}p_jAY)E\cos\beta + \epsilon^{\frac{4}{3}}\{[(-b_j^{\pm}r_j\log|Y| + d_j^{\pm}) \\ &+ A(a_j^{\pm}Y + b_j^{\pm}Y\log|Y|)]E\cos\beta z + b_j^{\pm}BE^2\} + \epsilon^{\frac{5}{3}}\log\epsilon^{\frac{1}{3}}\{[(a_j^{\pm}r_j + b_j^{\pm}s_j + d_j^{\pm}p_j)Y \\ &+ \tfrac{1}{2}Ap_jb_j^{\pm}Y^2]E\cos\beta z + (-b_j^{\pm}R_j + b_j^{\pm}p_jBY)E^2\} \\ &+ \epsilon^{\frac{5}{3}}\{[c_j^{\pm}Y + (a_j^{\pm}r_j + b_j^{\pm}s_j + d_j^{\pm}p_j)Y\log|Y|]E\cos\beta z + [(-b_j^{\pm}R_j\log|Y| + D_j^{\pm}) \\ &+ B(a_j^{\pm}Y + b_j^{\pm}Y\log|Y|)]E^2\} + \epsilon^{\frac{6}{3}}\log\epsilon^{\frac{1}{3}}\{[(a_j^{\pm}R_j + b_j^{\pm}S_j + D_j^{\pm}p_j)Y \\ &+ \tfrac{1}{2}Bp_jb_j^{\pm}Y^2]E^2 + \ldots\} + \epsilon^{\frac{6}{3}}\{[C_j^{\pm}Y + (a_j^{\pm}R_j + b_j^{\pm}S_j + D_j^{\pm}p_j)Y\log|Y|]E^2\} \\ &+ \text{c.c.} + \ldots, \end{aligned} \tag{3.34}$$

$$u \sim \epsilon^{\frac{2}{3}}(-\mathrm{i}\alpha)^{-1}\sin^2\theta Ab_j^{\pm}Y^{-1}E\cos\beta z + \text{c.c.} + \ldots, \tag{3.35}$$

$$w \sim \epsilon^{\frac{2}{3}}\bar{\alpha}^{-1}\sin\theta b_j^{\pm}AY^{-1}E\cos\beta z + \text{c.c.} + \ldots, \tag{3.36}$$

$$p \sim \epsilon\mathrm{i}\alpha^{-1}\bar{U}_y\cos\theta Ab_j^{\pm}E\cos\beta z + \epsilon^{\frac{4}{3}}\tfrac{1}{2}\mathrm{i}\alpha^{-1}\bar{U}_y b_j^{\pm}BE^2 + \text{c.c.} + \ldots. \tag{3.37}$$

3.2. *Solvability conditions*

Multiplying both sides of (3.17) by $\bar{v}_1$, integrating from 0 to $+\infty$, and using the asymptotic solutions (3.14) and (3.18), we obtain the solvability condition for (3.17):

$$\mathrm{i}\alpha^{-1}J_1\frac{\mathrm{d}A}{\mathrm{d}t_1}+J_2\tau_1 A = -\sum_j\{(b_j^+c_j^+-b_j^-c_j^-)-r_j(b_j^+a_j^+-b_j^-a_j^-) \\ -p_j(b_j^+d_j^+-b_j^-d_j^-)-(a_j^+d_j^+-a_j^-d_j^-)\}, \quad (3.38)$$

where the sum is over all critical layers, and J_1 and J_2 are constants defined by

$$J_1 = \int_0^{+\infty}\frac{\bar{U}_{yy}}{(\bar{U}-c)^2}\bar{v}_1^2\,\mathrm{d}y, \quad (3.39)$$

$$J_2 = \int_0^{+\infty}\left[-\frac{\bar{U}_{yy}\bar{U}_\tau}{(\bar{U}-c)^2}+\frac{\bar{U}_{yy\tau}}{(\bar{U}-c)}\right]\bar{v}_1^2\,\mathrm{d}y. \quad (3.40)$$

Note that these integrals are singular and should be interpreted in the sense of Hadamard (see §4).

By the same procedure, the solvability condition for (3.23) is derived as follows:

$$\mathrm{i}(2\alpha)^{-1}J_1\frac{\mathrm{d}B}{\mathrm{d}t_1}+J_2\tau_1 B = -\sum_j\{(b_j^+C_j^+-b_j^-C_j^-)-R_j(b_j^+a_j^+-b_j^-a_j^-) \\ -p_j(b_j^+D_j^+-b_j^-D_j^-)-(a_j^+D_j^+-a_j^-D_j^-)\}. \quad (3.41)$$

Equations (3.38) and (3.41) are crucial in deriving the amplitude equations. The jumps $(a_j^+-a_j^-)$ etc. will be determined by analysing the critical layers. It is important to note that the left-hand sides of (3.38) and (3.41) are linear and that nonlinear terms come from the critical layers through these jumps. Thus in the analysis of the critical layers, it is sufficient to concentrate only on the solutions contributing to these jumps. Since we find that the solution driven by the quadratic interaction of the three-dimensional waves does not contribute to any jump, the quadratic term involving the three-dimensional amplitude will not appear in the amplitude equation of the two-dimensional wave. The reader who is not especially concerned with the rather messy algebraic details within the critical layers can omit the following section at a first reading.

4. Inner expansion

Expressions (3.34)–(3.37) suggest that the inner expansions within the jth critical layer take the form:

$$u = \epsilon^{2/3}U_1+\epsilon^{3/3}U_2+\epsilon^{4/3}U_3+\epsilon^{5/3}U_4+\dots, \quad (4.1)$$

$$v = \epsilon V_1+\epsilon^{4/3}V_2+\epsilon^{5/3}V_3+\epsilon^{6/3}V_4+\dots, \quad (4.2)$$

$$w = \epsilon^{2/3}W_1+\epsilon^{3/3}W_2+\epsilon^{4/3}W_3+\epsilon^{5/3}W_4+\dots, \quad (4.3)$$

$$p = \epsilon P_1+\epsilon^{4/3}P_2+\epsilon^{5/3}P_3+\dots, \quad (4.4)$$

where the $O(\epsilon^n\log\epsilon^{1/3})$ terms are not written out explicitly since they match onto the outer solutions automatically whenever the solutions at $O(\epsilon^n)$ match with their corresponding outer expansions.

We shall see that the solutions at the first two orders are available from the analysis for a pair of oblique waves (Goldstein & Choi 1989; Wu 1991), but with some modification due to the presence of the two-dimensional disturbance.

The function V_1 satisfies the equation

$$\mathrm{L}_0 \partial^2 V_1/\partial Y^2 = 0, \tag{4.5}$$

where

$$\mathrm{L}_0 = \partial/\partial t_1 + (\bar{U}_y Y + \bar{U}_\tau \tau_1)\,\partial/\partial x. \tag{4.6}$$

The solution

$$V_1 = \hat{A}(t_1) E \cos\beta z + \text{c.c.} \tag{4.7}$$

matches with the outer solution if $\hat{A}(t_1) = b_j A(t_1)$, and $b_j^+ = b_j^- = b_j$, i.e. if the jump $(b_j^+ - b_j^-)$ is zero.

Expansion of the y-momentum equation gives

$$\partial P_1/\partial Y = 0, \tag{4.8}$$

and so

$$P_1 = \mathrm{i}\alpha^{-1}\bar{U}_y \cos\theta \hat{A} E \cos\beta z + \text{c.c.}$$

The function W_1 satisfies the equation

$$\mathrm{L}_0 W_1 = -\partial P_1/\partial z. \tag{4.9}$$

Let $W_1 = \hat{W}_1(Y, t_1) E \sin\beta z + \text{c.c.}$, then $\hat{W}_1$ satisfies

$$\hat{\mathrm{L}}_0^{(1)} \hat{W}_1 = \mathrm{i}\bar{U}_y \sin\theta\cos\theta \hat{A},$$

where

$$\hat{\mathrm{L}}_0^{(n)} = \partial/\partial t_1 + n\mathrm{i}\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1).$$

The solution that matches with the outer solution is

$$\hat{W}_1 = \mathrm{i}\bar{U}_y \sin\theta\cos\theta \hat{W}_0^{(0)},$$

where we have put

$$\hat{W}_0^{(n)} = \int_0^{+\infty} \xi^n \hat{A}(t_1 - \xi)\,\mathrm{e}^{-\mathrm{i}\Omega\xi}\,\mathrm{d}\xi, \quad \Omega = \alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1). \tag{4.10}$$

Similarly, we write $U_1 = \hat{U}_1(Y, t_1) E \cos\beta z + \text{c.c.}$, and then

$$\hat{U}_1 = -\bar{U}_y \sin^2\theta \hat{W}_0^{(0)}.$$

At $O(\epsilon^{\frac{4}{3}})$, we write the pressure as

$$P_2 = \hat{P}_2^{(2,0)}(Y, t_1) E^2 + \hat{P}_2^{(1,1)} E \cos\beta z + \text{c.c.},$$

where $\hat{P}_2^{(2,0)}$ satisfies

$$\partial \hat{P}_2^{(2,0)}/\partial Y = 0. \tag{4.11}$$

The solution that matches with the outer expansion is

$$\hat{P}_2^{(2,0)} = \tfrac{1}{2}\mathrm{i}\alpha^{-1}\bar{U}_y \hat{B}. \tag{4.12}$$

The three-dimensional pressure component $\hat{P}_2^{(1,1)}$ is not needed in the following analysis.

The velocity V_2 satisfies

$$\mathrm{L}_0 V_{2,YY} = \mathrm{L}_1 V_1 + \frac{\partial}{\partial Y}\left[\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{31}}{\partial z}\right], \tag{4.13}$$

where $$\mathrm{L}_1 = -(\tfrac{1}{2}\bar{U}_{yy}Y^2+\bar{U}_{y\tau}\tau_1 Y+\tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2)\frac{\partial^3}{\partial x\,\partial Y^2}+\bar{U}_{yy}\frac{\partial}{\partial x}, \tag{4.14}$$

and $$S_{11}=\frac{\partial U_1^2}{\partial x}+\frac{\partial U_1 V_1}{\partial Y}+\frac{\partial U_1 W_1}{\partial z},\quad S_{31}=\frac{\partial U_1 W_1}{\partial x}+\frac{\partial V_1 W_1}{\partial Y}+\frac{\partial W_1^2}{\partial z}.$$

Furthermore, the Reynolds stresses S_{11} and S_{31} can be expressed as

$$S_{11}=S_{11}^{(0,0)}+S_{11}^{(0,2)}\cos 2\beta z+S_{11}^{(2,0)}E^2+S_{11}^{(2,2)}E^2\cos 2\beta z+\text{c.c.}, \tag{4.15}$$

$$S_{31}=S_{31}^{(0,2)}\cos 2\beta z+S_{31}^{(2,2)}E^2\cos 2\beta z+\text{c.c.} \tag{4.16}$$

After some calculation, it can be shown that

$$S_{11}^{(0,0)}=\tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta\hat{A}^*\hat{W}_0^{(1)}, \tag{4.17}$$

$$S_{11}^{(0,2)}=\tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta\hat{A}^*\hat{W}_0^{(1)}, \tag{4.18}$$

$$S_{11}^{(2,0)}=\tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta[\hat{A}\hat{W}_0^{(1)}+2\sin^2\theta\hat{W}_0^{(0)}\hat{W}_0^{(0)}], \tag{4.19}$$

$$S_{11}^{(2,2)}=\tfrac{1}{2}\mathrm{i}\alpha\bar{U}_y^2\sin^2\theta\hat{A}\hat{W}_0^{(1)}, \tag{4.20}$$

$$S_{31}^{(0,2)}=\tfrac{1}{2}\beta\bar{U}_y^2\cos^2\theta[\hat{A}^*\hat{W}_0^{(1)}+2\sin^2\theta\hat{W}_0^{(0)}\hat{W}_0^{*(0)}], \tag{4.21}$$

$$S_{31}^{(2,2)}=\tfrac{1}{2}\beta\bar{U}_y^2\cos^2\theta\hat{A}\hat{W}_0^{(1)}. \tag{4.22}$$

Inspecting the right-hand side of (4.13), and using (4.15) and (4.16), we conclude that V_2 has the solution of the following form

$$V_2=\hat{V}_2^{(1)}E\cos\beta z+\hat{V}_2^{(0,2)}\cos 2\beta z+\hat{V}_2^{(2,0)}E^2+\text{c.c.} \tag{4.23}$$

The three-dimensional fundamental component $\hat{V}_2^{(1)}$ is driven by the linear forcing term only, i.e. $\mathrm{L}_1 v_1=\mathrm{i}\alpha\bar{U}_{yy}\hat{A}$, which is exactly the same as in the two-dimensional case studied by Wu (1991). By the same procedure, we obtain the following jump conditions:

$$a_j^+-a_j^-=\pi i p_j b_j \operatorname{sgn}(\bar{U}_y), \tag{4.24}$$

$$d_j^+-d_j^-=-\pi i r_j b_j \operatorname{sgn}(\bar{U}_y). \tag{4.25}$$

Substituting (4.17)–(4.22) into (4.13), we find that

$$(\partial/\partial t_1)\,\hat{V}_{2,YY}^{(0,2)}=-\mathrm{i}\bar{S}^3\sin^2\theta[\hat{A}^*\hat{W}_0^{(2)}+4\sin^2\theta\hat{W}_0^{*(0)}\hat{W}_0^{(1)}], \tag{4.26}$$

$$\hat{\mathrm{L}}_0^{(2)}\,\hat{V}_{2,YY}^{(2,0)}=\mathrm{i}\bar{S}^3\sin^2\theta[\hat{A}\hat{W}_0^{(2)}+4\sin^2\theta\hat{W}_0^{(0)}\hat{W}_0^{(1)}], \tag{4.27}$$

where we have put

$$\bar{S}=\alpha\bar{U}_y. \tag{4.28}$$

The solutions are found to be

$$\hat{V}_{2,YY}^{(0,2)}=-\mathrm{i}\bar{S}^3\sin^2\theta\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta\,\xi(\xi+4\sin^2\theta\eta)\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\Omega\xi}, \tag{4.29}$$

$$\hat{V}_{2,YY}^{(2,0)}=\mathrm{i}\bar{S}^3\sin^2\theta\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta[\xi^2+4\sin^2\theta(\xi+\eta)\eta]\hat{A}(t_1-\eta)\,\hat{\mathrm{A}}(\mathrm{t}_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\eta)}. \tag{4.30}$$

In writing $\hat{V}_{2,YY}^{(0,2)}$ in the compact form (4.29), we have omitted a purely imaginary part; from (4.23) this does not alter the physical velocity. A similar procedure will be followed on simplifying the solution for $\hat{W}^{(0,2)}$ later on. Integrating $\hat{V}^{(2,0)}_{2,YY}$ once with respect to Y yields

$$\hat{V}^{(2,0)}_{2,Y} = -\bar{S}^2 \sin^2\theta \int_0^{+\infty} \mathrm{d}\xi \int_0^{+\infty} \mathrm{d}\eta\, [\xi^2 + 4\sin^2\theta(\xi+\eta)\eta]\,(\xi+2\eta)^{-1}\,[\mathrm{e}^{-\mathrm{i}\bar{\Omega}(\xi+2\eta)} - 1] \times \hat{A}(t_1-\eta)\,\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(\xi+2\eta)}, \quad (4.31)$$

where we have put

$$\bar{\Omega} = \alpha \bar{U}_y Y, \quad \bar{\omega} = \alpha \bar{U}_\tau \tau_1, \quad (4.32)$$

and an integration constant (or more precisely a function of t_1) is taken to be zero by matching with the corresponding outer solution.

Instead of integrating $\hat{V}^{(2,0)}_{2,Y}$ to get $\hat{V}^{(2,0)}_2$, we obtain it from the x-momentum equation together with the continuity equation in the following way. The continuity equation is

$$2\mathrm{i}\alpha \hat{U}^{(2,0)}_2 + \hat{V}^{(2,0)}_{2,Y} = 0, \quad (4.33)$$

and the x-momentum equation is

$$\hat{\mathrm{L}}^{(2)}_0 \hat{U}^{(2,0)}_2 + \bar{U}_y \hat{V}^{(2,0)}_2 = -(2\mathrm{i}\alpha)\hat{P}^{*(2,0)}_2 - S^{(2,0)}_{11}. \quad (4.34)$$

The first term of the right-hand side, which is absent in the case of a pair of oblique waves (Goldstein & Choi 1989; Wu 1991), is the pressure related to the two-dimensional wave, and the second term is the forcing from the nonlinear interaction. From the above two equations we obtain

$$\hat{V}^{(2,0)}_2 = \hat{B} - \tfrac{1}{2}\mathrm{i}\bar{S}^{-1}\hat{\mathrm{L}}^{(2)}_0 \hat{V}^{(2,0)}_{2,Y} - \bar{U}_y^{-1} S^{(2,0)}_{11}. \quad (4.35)$$

It transpires that the above expression for $\hat{V}^{(2,0)}_2$ is helpful in evaluating the asymptotic behaviour of related solutions at higher order.

The function W_2 satisfies the following equation:

$$\mathrm{L}_0 W_2 = -\partial P_2/\partial z - F_2(Y)\,\partial W_1/\partial x - S_{31}, \quad (4.36)$$

where

$$F_2(Y) = \tfrac{1}{2} - \bar{U}_{yy} Y^2 + \bar{U}_{y\tau}\tau_1 Y + \tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2.$$

The velocity W_2 has the form

$$W_2 = \hat{W}^{(1)}_2 E \sin\beta z + \hat{W}^{(0,2)}_2 \sin 2\beta z + \hat{W}^{(2,2)}_2 E^2 \sin 2\beta z + \text{c.c.}, \quad (4.37)$$

where $\hat{W}^{(1)}_2$ is the component driven by linear forcing through the first two terms on the right-hand side of (4.36); it does not need to be worked out because it does not contribute to the jumps. The solutions forced by the nonlinear interaction are $\hat{W}^{(0,2)}_2$ and $\hat{W}^{(2,2)}_2$, which satisfy the following equations respectively:

$$(\partial/\partial t_1)\,\hat{W}^{(0,2)}_2 = -S^{(0,2)}_{31}, \quad (4.38)$$

$$\hat{\mathrm{L}}^{(2)}_0 \hat{W}^{(2,2)}_2 = -S^{(2,2)}_{31}. \quad (4.39)$$

The solutions are

$$\hat{W}^{(0,2)}_2 = -\tfrac{1}{2}\beta^{-1}\bar{S}^2\sin^2\theta \int_0^{+\infty}\mathrm{d}\xi \int_0^{+\infty}\mathrm{d}\eta\,(\xi + 4\sin^2\theta\,\eta)\,\hat{A}^*(t_1-\eta)\,\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\Omega\xi}, \quad (4.40)$$

$$\hat{W}^{(2,2)}_2 = -\tfrac{1}{2}\beta^{-1}\bar{S}^2\sin^2\theta \int_0^{+\infty}\mathrm{d}\xi \int_0^{+\infty}\mathrm{d}\eta\,\xi\hat{A}(t_1-\eta)\,\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\eta)}. \quad (4.41)$$

Hence from the continuity equation $\hat{V}^{(0,2)}_{2,Y}+2\beta\hat{W}^{(0,2)}_2=0$, we have that

$$\hat{V}^{(0,2)}_{2,Y}=\bar{S}^2\sin^2\theta\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta(\xi+4\sin^2\theta\eta)\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\Omega\xi}. \quad (4.42)$$

Integration with respect to Y then yields

$$\hat{V}^{(0,2)}_2=\mathrm{i}\bar{S}\sin^2\theta\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta\xi^{-1}(\xi+4\sin^2\theta\eta)\,[\mathrm{e}^{-\mathrm{i}\bar{\Omega}\xi}-1]$$
$$\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\bar{\omega}\xi}+\hat{V}^{(0,2)}_2(0,t_1). \quad (4.43)$$

Note that as $\bar{\Omega}\rightarrow\pm\infty$, i.e. $Y\rightarrow\pm\infty$ (see (4.32)),

$$(\hat{V}^{(0,2)}_2+\text{c.c.})\sim\pm4\bar{S}\sin^4\theta\pi\int_0^{+\infty}\eta|\hat{A}(t_1-\eta)|^2\,\mathrm{d}\eta+[\text{order-one 'no-jump' terms}]+o(1).$$

We conclude that a vortex component must be included in the $O(\epsilon^{\frac{4}{3}})$ outer expansion of v so that it matches to $\hat{V}^{(0,2)}_2$.

The function U_2 satisfies

$$\mathrm{L}_0U_{2,Y}=-\bar{U}_{yy}V_1+F_1(Y)\frac{\partial W_1}{\partial z}-F_2(Y)\frac{\partial^2U_1}{\partial x\,\partial Y}-\frac{\partial S_{11}}{\partial Y}+\bar{U}_y\frac{\partial W_2}{\partial z}, \quad (4.44)$$

where
$$F_1(Y)=\bar{U}_{yy}Y+\bar{U}_{Y\tau}\tau_1.$$

The solution has the form

$$U_2=\hat{U}^{(1)}_2E\cos\beta z+\hat{U}^{(0,0)}_2+\hat{U}^{(0,2)}_2\cos2\beta z+\hat{U}^{(2,0)}_2E^2+\hat{U}^{(2,2)}_2E^2\cos2\beta z+\text{c.c.} \quad (4.45)$$

Again only the second harmonic and the mean-flow distortion induced by nonlinearity need to be calculated. From the continuity equation, we obtain

$$\hat{U}^{(2,0)}_2=-(2\mathrm{i}\alpha)^{-1}\hat{V}^{(2,0)}_{2,Y},\quad \hat{U}^{(2,2)}_2=\mathrm{i}\beta\alpha^{-1}\hat{W}^{(2,2)}_2.$$

The terms representing the mean-flow distortion satisfy the following equations:

$$(\partial/\partial t_1)\,\hat{U}^{(0,0)}_{2,Y}=-\tfrac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta\hat{A}^*\hat{W}^{(2)}_0, \quad (4.46)$$

$$(\partial/\partial t_1)\,\hat{U}^{(0,2)}_{2,Y}=-(\partial/\partial Y)\,S^{(0,2)}_{11}+2\beta\bar{U}_y\,\hat{W}^{(0,2)}_2. \quad (4.47)$$

The solutions are

$$\hat{U}^{(0,0)}_{2,Y}=-\tfrac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta\xi^2\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\Omega\xi}, \quad (4.48)$$

$$\hat{U}^{(0,2)}_{2,Y}=-\tfrac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\mathrm{d}\xi\int_0^{+\infty}\mathrm{d}\eta[\xi^2+2\xi\eta+4\sin^2\theta\eta^2]\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\mathrm{i}\Omega\xi}. \quad (4.49)$$

Integration once with respect to Y yields

$$\hat{U}^{(0,2)}_2=-\tfrac{1}{2}\mathrm{i}\alpha^{-1}\bar{S}^2\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\xi^{-1}[\xi^2+2\xi\eta+4\sin^2\theta\eta^2]\,[\mathrm{e}^{-\mathrm{i}\bar{\Omega}\xi}-1]$$
$$\times\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\,\mathrm{e}^{-\bar{\omega}\xi}\mathrm{d}\xi\,\mathrm{d}\eta+\hat{U}^{(0,2)}_2(0,t_1). \quad (4.50)$$

As $Y\rightarrow\pm\infty$,

$$(\hat{U}^{(0,2)}_2+\text{c.c.})\sim\mp4\mathrm{i}\alpha^{-1}\bar{S}\sin^4\theta\pi\int_0^{+\infty}\eta^2|\hat{A}(t_1-\eta)|^2\mathrm{d}\eta$$
$$+\{\text{order-one 'no-jump' terms}\}+o(1),$$

which is in fact the streamwise slip velocity generated by the nonlinear interaction inside the critical layers, to which the outer expansion of u has to match. This is the reason why the leading-order outer expansion of u should contain a spanwise-dependent mean-flow. Note that this mean-flow is strong in the sense that it has the same magnitude as the three-dimensional waves. We will show that the interaction of the fundamental three-dimensional waves with this mean flow affects the evolution of the waves. The significance of mean-flow/wave interactions has been studied extensively, e.g. see Benney (1984), Hall & Smith (1989, 1990, 1991), Smith & Walton (1990). In particular, Hall & Smith (1991) have shown that the nonlinear interaction of small-amplitude waves may produce an order-one mean-flow distortion, thus completely altering the mean-flow profile from its original laminar state. Compared with the present study, their critical layers are predominantly steady and viscous. This is because their nonlinear evolution timescale is much slower, with the result that only very small-amplitude waves are needed to generate a strong vortex flow (see Wu *et al.* 1992 for a further discussion of this point).

We first seek a solution for V_3; this is found to satisfy the following equation:

$$\mathrm{L}_0 V_{3,YY} = \mathrm{L}_1 V_2 + \mathrm{L}_2 V_1 + \frac{\partial}{\partial Y}\left[\frac{\partial}{\partial x}S_{12} + \frac{\partial}{\partial z}S_{32}\right], \tag{4.51}$$

where L_1 is defined by (4.14),

$$\begin{aligned}\mathrm{L}_2 = & -[\tfrac{1}{6}\mathrm{U}_{yyy}Y^3 + \bar{U}_{yy\tau}\tau_1 Y^2 + \bar{U}_{y\tau\tau}\tau_1^2 Y + \tfrac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3]\frac{\partial^3}{\partial x\,\partial Y^2} \\ & + [\bar{U}_{yyy}Y + \bar{U}_{yy\tau}\tau_1]\frac{\partial}{\partial x} - \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1)\frac{\partial}{\partial x}\right]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right],\end{aligned} \tag{4.52}$$

and the Reynolds stresses S_{12} and S_{32} are given by

$$S_{12} = (\partial/\partial x)(2U_1 U_2) + (\partial/\partial Y)(U_1 V_2 + U_2 V_1) + (\partial/\partial z)(U_1 W_2 + U_2 W_1), \tag{4.53}$$

$$S_{32} = (\partial/\partial x)(U_1 W_2 + U_2 W_1) + (\partial/\partial Y)(W_1 V_2 + W_2 V_1) + (\partial/\partial z)(2W_1 W_2). \tag{4.54}$$

The Reynolds stresses contain different components, among which only those proportional to $E\cos\beta z$ and $E^3\cos\beta z$ will be needed later. Therefore we write

$$S_{12} = S_{12}^{(1,1)}E\cos\beta z + S_{12}^{(3,1)}E^3\cos\beta z + \text{c.c.} + \dots, \tag{4.55}$$

$$S_{32} = S_{32}^{(1,1)}E\sin\beta z + S_{32}^{(3,1)}E^3\sin\beta z + \text{c.c.} + \dots. \tag{4.56}$$

After some calculation, we have

$$\begin{aligned}S_{12}^{(1,1)} = & \hat{A}\hat{U}_{2,Y}^{*(0,0)} + \tfrac{1}{2}\hat{A}\hat{U}_{2,Y}^{*(0,2)} - \beta\hat{W}_1\hat{U}_2^{*(0,0)} + \tfrac{1}{4}\hat{U}_1\hat{V}_{2,Y}^{*(0,2)} \\ & + [\tfrac{3}{2}\mathrm{i}\alpha\hat{U}_1\hat{U}_2^{*(0,2)} + \tfrac{1}{2}\hat{U}_{1,Y}\hat{V}_2^{*(0,2)}] + [\hat{U}_{1,Y}^*\hat{V}_2^{(2,0)} - \tfrac{1}{2}\hat{U}_1^*\hat{V}_{2,Y}^{(2,0)}],\end{aligned} \tag{4.57}$$

$$S_{12}^{(3,1)} = \hat{A}\hat{U}_{2,Y}^{(2,0)} + \tfrac{1}{2}\hat{A}\hat{U}_{2,Y}^{(2,2)} - 3\beta\hat{U}_1\hat{W}_2^{(2,2)} + [\hat{U}_{1,Y}\hat{V}_2^{(2,0)} - \tfrac{3}{2}\hat{U}_1\hat{V}_{2,Y}^{(2,0)}], \tag{4.58}$$

$$\begin{aligned}S_{32}^{(1,1)} = & [\tfrac{1}{2}\hat{A}\hat{W}_{2,Y}^{*(0,2)} + \mathrm{i}\alpha\hat{W}_1\hat{U}_2^{*(0,0)} - \tfrac{1}{2}\beta\hat{W}_1\hat{W}_2^{*(0,2)}] \\ & + [-\tfrac{1}{2}\mathrm{i}\alpha\hat{W}_1\hat{U}_2^{*(0,2)} - \tfrac{1}{2}\hat{W}_{1,Y}\hat{V}_2^{*(0,2)}] + [\tfrac{1}{2}\hat{W}_1^*\hat{V}_{2,Y}^{(2,0)} + \hat{W}_{1,Y}^*\hat{V}_2^{(2,0)}],\end{aligned} \tag{4.59}$$

$$S_{32}^{(3,1)} = \tfrac{1}{2}\hat{A}\hat{W}_{2,Y}^{(2,2)} - \beta\hat{W}_1\hat{W}_2^{(2,2)} - \tfrac{1}{2}\hat{W}_1\hat{V}_{2,Y}^{(2,0)} + \hat{W}_{1,Y}\hat{V}_2^{(2,0)}. \tag{4.60}$$

Inspecting the right-hand side of (4.51) together with (4.55) and (4.56), we find that V_3 has the form

$$V_3 = \hat{V}_3 E\cos\beta z + \tilde{V}_3 E^3\cos\beta z + \hat{V}_3^{(2,0)}E^2 + \text{c.c.} + \dots, \tag{4.61}$$

where the component proportional to $E\cos 3\beta z$ has been omitted since it will not be used in deriving the evolution equations. However, to derive the amplitude equation for the two-dimensional wave, we need to solve for two more components, namely, $\tilde{V}_3 E^3 \cos\beta z$ and $\hat{V}_3^{(2,0)} E^2$. Thus we write the derivatives of the Reynolds stresses as follows:

$$(\partial/\partial x)\, S_{12} + (\partial/\partial z)\, S_{32} = \bar{M} E \cos\beta z + \tilde{M} E^3 \cos\beta z + \hat{M}^{(2,0)} E^2 + \text{c.c.} + \dots.$$

It is noted that the relevant part of the linear forcing term, i.e. $(\mathrm{L}_1 V_2 + \mathrm{L}_2 V_1)$, is the same as $F^{(l)}(Y, t_1)E$ in the two-dimensional case (Wu 1991). So the solution (denoted here by $V_3^{(l)}E$) driven by it has the following asymptotic behaviour as $Y \to \pm\infty$,

$$\hat{V}_{3,Y}^{(l)} \sim (a_j^+ p_j + 2q_j b_j + \tfrac{1}{2} p_j^2 b_j)\, Y + (a_j^+ r_j + p_j d_j^+ + s_j b_j)\log|Y| \\ + \{\pm \tfrac{1}{2}\pi \mathrm{i}\, \operatorname{sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) + \dots\}. \quad (4.62)$$

We now write

$$\bar{M} = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_0,$$

where
$$\bar{M}_1 = \mathrm{i}\alpha \hat{A} \hat{U}_2^{*(0,0)} + \tfrac{1}{2}\mathrm{i}\alpha \hat{A} \hat{U}_{2,Y}^{*(0,2)} + \tfrac{1}{2}\beta \hat{A} \hat{W}_{2,Y}^{*(0,2)}, \quad (4.63)$$

$$\bar{M}_2 = \mathrm{i}\alpha \hat{U}_{1,Y} \hat{V}_2^{*(0,2)} - 2\alpha^2 \hat{U}_1 \hat{U}_2^{*(0,2)}, \quad (4.64)$$

$$\bar{M}_3 = 2\mathrm{i}\alpha \hat{U}_{1,Y}^{*} \hat{V}_2^{(2,0)}, \quad (4.65)$$

$$\bar{M}_0 = \mathrm{i}\alpha \hat{A} \hat{U}_{2,Y}^{(0,0)} + \tfrac{1}{2}\mathrm{i}\alpha \hat{A} \hat{U}_{2,Y}^{(0,2)} + \tfrac{1}{2}\beta \hat{A} \hat{W}_{2,Y}^{(0,2)} + \mathrm{i}\alpha \hat{A}^* \hat{U}_2^{(2,0)} \\ + \tfrac{1}{2}\alpha \hat{A}^* \hat{U}_2^{(2,2)} + \tfrac{1}{2}\beta \hat{A}^* \hat{W}_2^{(2,2)} + \mathrm{i}\alpha \hat{U}_{1,Y} \hat{V}_2^{(0,2)} - 2\alpha^2 \hat{U}_1 \hat{U}_2^{(0,2)}.$$

The nonlinear forcing terms are deliberately split as above in order to aid the calculation of the solutions and their asymptotic behaviour. The solutions of (4.51) driven by $\bar{M}_{j,Y}$ are denoted by $\hat{V}_3^{(j)}$ $(j = 0,1,2,3)$ respectively, i.e.

$$\hat{\mathrm{L}}_0^{(1)} \hat{V}_{3,YY}^{(j)} = \bar{M}_{j,Y} \quad (j = 0,1,2,3); \quad (4.66)$$

while
$$\hat{V}_3 = \hat{V}_3^{(l)} + \hat{V}_3^{(1)} + \hat{V}_3^{(2)} + \hat{V}_3^{(3)} + \hat{V}_3^{(0)}.$$

We find that $\hat{V}_3^{(0)}$ makes no contribution to the jumps, and so matching $\hat{V}_{3,Y} = (\hat{V}_{3,Y}^{(l)} + \hat{V}_{3,Y}^{(1)} + \hat{V}_{3,Y}^{(2)} + \hat{V}_{3,Y}^{(3)})$ with the outer solution will yield the jump $(c_j^+ - c_j^-)$ – see (4.79).

We now solve for $\hat{V}_{3,YY}^{(j)}$ $(j = 1,2,3)$. It can be shown that

$$\hat{V}_{3,YY}^{(1)} = \tfrac{1}{2}\bar{S}^4 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \bar{K}_v^{(1)}(\xi,\eta) \\ \hat{A}(t_1 - \zeta)\hat{A}(t_1\eta - \zeta) \times \hat{A}^*(t_1 - \xi - \eta - \zeta)\, \mathrm{e}^{-\mathrm{i}\Omega(\zeta - \xi)} \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta, \quad (4.67)$$

where
$$\bar{K}_v^{(1)}(\xi,\eta) = 2\xi^3 + \xi^2\eta + 2\sin^2\theta(\xi^2\eta + \xi\eta^2). \quad (4.68)$$

Integrating with respect to Y from $-\infty$ to $+\infty$, we obtain the jump

$$\hat{V}_{3,Y}^{(1)}(+\infty) - \hat{V}_{3,Y}^{(1)}(-\infty) \\ = \bar{S}^4 j_0 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} \bar{K}_v^{(1)}(\xi,\eta)\hat{A}(t_1 - \xi)\hat{A}(t_1 - \eta - \xi)\hat{A}^*(t_1 - 2\xi - \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad (4.69)$$

where we have put

$$j_0 = \pi|\bar{S}|^{-1}. \quad (4.70)$$

The solution driven by $\bar{M}_{2,Y}$ needs more effort. After some algebraic manipulation, we find

$$\frac{\partial \bar{M}_2}{\partial Y} = \bar{S}^4 \sin^4\theta \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi^{-1}[\xi^2+\xi\zeta+2\xi\eta+4\sin^2\theta(\eta+\zeta)\eta]\,[(\zeta-\xi)\,e^{-i\bar{\Omega}(\zeta-\xi)}-\zeta\, e^{-i\bar{\Omega}\zeta}]$$
$$\times \hat{A}(t_1-\zeta)\,\hat{A}(t_1-\eta)\,\hat{A}^*(t_1-\eta-\xi)\,e^{-i\bar{\omega}(\zeta-\xi)}\,d\xi\, d\eta\, d\zeta.$$

The solution induced by it can be expressed as

$$\hat{V}^{(2)}_{3,YY} = \bar{S}^4 \sin^4\theta J, \tag{4.71}$$

where J, and the necessary algebraic manipulations to obtain the jump of $\hat{V}^{(2)}_{3,Y}$, are given in Appendix A. Using (A 2), (A 7), (4.71) and (A 1), we find that

$$\hat{V}^{(2)}_{3,Y}(+\infty)-\hat{V}^{(2)}_{3,Y}(-\infty)$$
$$= \bar{S}^4 j_0 \sin^4\theta \int_0^{+\infty}\int_0^{+\infty} \bar{K}^{(2)}_v(\xi,\eta)\,\hat{A}(t_1-\xi)\,\hat{A}(t_1-\eta-\xi)\,\hat{A}^*(t_1-2\xi-\eta)\,d\xi\, d\eta, \tag{4.72}$$

where
$$\bar{K}^{(2)}_v(\xi,\eta) = -4\xi^2(\xi+\eta)+4\sin^2\theta[-\tfrac{2}{3}\xi^3-\xi^2\eta-\xi\eta^2]. \tag{4.73}$$

The forcing $\bar{M}_{3,Y}$ is the Reynolds stress generated by the interaction between the three-dimensional fundamental and the two-dimensional wave component. Using (4.35) we can show that

$$\partial\bar{M}_3/\partial Y = 2i\alpha\hat{B}\hat{U}^*_{1,YY}+\alpha\bar{S}^{-1}\hat{U}^*_{1,YY}\,\mathrm{L}^{(2)}_0\hat{V}^{(2,0)}_{2,Y}+2i\alpha\hat{U}^*_{1,Y}\,\hat{V}^{(2,0)}_{2,Y}-2i\alpha\hat{U}^{-1}_y\,\hat{U}^*_{1,YY}S^{(2,0)}_{11}. \tag{4.74}$$

We find that the solution for $\hat{V}^{(3)}_{3,YY}$ is

$$\hat{V}^{(3)}_{3,YY} = 2i\alpha\int_{-\infty}^{t_1} d\tilde{t}\, e^{-i\Omega(t_1-\tilde{t})}\,\hat{B}(\tilde{t})\,\hat{U}^*_{1,YY}(\tilde{t})$$
$$+\alpha\bar{S}^{-1}\hat{U}^*_{1,YY}\,\hat{V}^{(2,0)}_{2,Y}-2i\alpha\bar{U}^{-1}_y\int_{-\infty}^{t_1} d\tilde{t}\, e^{-i\Omega(t_1-\tilde{t})}\,\hat{U}^*_{1,YY}(\tilde{t})\,S^{(2,0)}_{11}(\tilde{t}).$$

For brevity, we write $\hat{V}^{(3)}_{3,YY}$ as

$$\hat{V}^{(3)}_{3,YY} = 2i\bar{S}^3\sin^2\theta I^{(b)}_v+\bar{S}^4\sin^4\theta I_v+\bar{S}^4\sin^4\theta I_1+2\bar{S}^4\sin^6\theta I_2, \tag{4.75}$$

where the expressions for $I^{(b)}_v$, I_v, I_1 and I_2, and the necessary analysis to obtain the jump of $\hat{V}^{(3)}_{3,Y}$, are given in Appendix B. Using (B 2), (B 3), (B 4), (B 5) and (4.75), we obtain the jump condition

$$\hat{V}^{(3)}_{3,Y}(+\infty)-\hat{V}^{(3)}_{3,Y}(-\infty) = 4i\bar{S}^3 j_0\sin^2\theta\int_0^{+\infty}\xi^2\hat{A}^*(t_1-2\xi)\,\hat{B}(t_1-\xi)\,d\xi$$
$$+\bar{S}^4 j_0\sin^4\theta\int_0^{+\infty}\int_0^{+\infty}\bar{K}^{(3)}_v(\xi,\eta)\,\hat{A}(t_1-\xi)\,\hat{A}(t_1-\eta-\xi)\,\hat{A}^*(t_1-2\xi-\eta)\,d\xi\,d\eta, \tag{4.76}$$

where
$$\bar{K}^{(3)}_v(\xi,\eta) = 2\xi^2\eta+4\sin^2\theta[\tfrac{2}{3}\xi^3]. \tag{4.77}$$

From matching $\hat{V}_{3,Y}$ to the outer expansion, we find that

$$c^+_j-c^-_j = \hat{V}_{3,Y}(+\infty)-\hat{V}_{3,Y}(-\infty), \tag{4.78}$$

and so using (4.62), (4.69), (4.72) and (4.76), we have that

$$c_j^+ - c_j^- = \pi \mathrm{i}\,\mathrm{sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) + \pi|\bar{S}|^3 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} K(\xi,\eta)\hat{A}(t_1-\xi)\hat{A}(t_1-\xi-\eta)\hat{A}^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + 4\pi\mathrm{i}\bar{S}^2\sin^2\theta\int_0^{+\infty}\xi^2\hat{A}^*(t_1-2\xi)\hat{B}(t_1-\xi)\,\mathrm{d}\xi, \tag{4.79}$$

where
$$K(\xi,\eta) = (2\xi^3+\xi^2\eta) - 2\sin^2\theta(2\xi^3-\xi\eta^2) - 4\sin^4\theta(\xi^2\eta+\xi\eta^2). \tag{4.80}$$

It is noted that $(a_j^+ - a_j^-)$ and $(d_j^+ - d_j^-)$ correspond to the classic $\pm\pi$ phase shift in the outer expansion, while $(c_j^+ - c_j^-)$ is modified by nonlinearity. As will be shown below, this has a significant effect on the evolution of the disturbance.

So far the jumps obtained are sufficient to derive the amplitude equation for the oblique waves. To derive the amplitude equation for the planar wave, we need to seek the jumps $(D_j^+ - D_j^-)$ and $(C_j^+ - C_j^-)$. The jump $(D_j^+ - D_j^-)$ can be obtained by solving for $\hat{V}_3^{(2,0)}$ (see (4.61)); this satisfies the equation

$$\hat{\mathrm{L}}_0^{(2)}\hat{V}_{3,YY}^{(2,0)} = \hat{\mathrm{L}}_1^{(2)}\hat{V}_3^{(2,0)} + \hat{M}_Y^{(2,0)}, \tag{4.81}$$

where
$$\hat{M}^{(2,0)} = 2\mathrm{i}\alpha\hat{U}_1\hat{U}_2^{(1)} + (\partial/\partial Y)[\tfrac{1}{2}\hat{U}_1\hat{V}_2^{(1)} + \tfrac{1}{2}\hat{A}\hat{U}_2^{(1)}].$$

Substitution of (4.35) into (4.81) yields

$$\hat{\mathrm{L}}_0^{(2)}\hat{V}_{3,YY}^{(2,0)} = \hat{\mathrm{L}}_1^{(2)}\hat{B} - 2\mathrm{i}\alpha F_2(Y)\hat{V}_{2,YY} + 2\mathrm{i}\alpha\bar{U}_{yy}[-\tfrac{1}{2}\mathrm{i}\bar{S}^{-1}\hat{\mathrm{L}}_0^{(2)}\hat{V}_{2,Y}^{(2,0)} - \bar{U}_y^{-1}S_{11}] + \hat{M}_Y^{(2,0)}. \tag{4.82}$$

The last three terms of the right-hand side are produced by the interaction between the oblique waves. After a tedious calculation it is found that the solution driven by them makes no contribution to the jump (4.83). The first forcing term is analogous to that in the two-dimensional case, and the solution driven by it therefore produces equivalent jumps (Wu 1991), namely

$$a_j^+ - a_j^- = \pi\mathrm{i}p_j b_j\,\mathrm{sgn}(\bar{U}_y), \quad D_j^+ - D_j^- = -\pi\mathrm{i}R_j b_j\,\mathrm{sgn}(\bar{U}_y). \tag{4.83}$$

Note that the first jump simply confirms the result obtained already.

The remaining jump condition to be determined is $(C_j^+ - C_j^-)$. This is given by the solution at the next order, i.e. V_4 (see (4.2)). To solve for V_4, we need to know the harmonic components $\tilde{V}_3$ and W_3. Before proceeding to seek solutions for them, we first simplify $\tilde{V}_{3,YY}$. Here we note that as far as deriving the amplitude equation for the two-dimensional wave is concerned, we need to concentrate only on $\hat{V}_{3,YY}^{(j)}$ ($j = 1,2,3,$) (see (4.66)); we need to consider $\hat{V}_{3,YY}^{(l)}$ and $\hat{V}_{3,YY}^{(0)}$ no further because they do not affect $(C_j^+ - C_j^-)$.

By means of the transformation $(\eta-\zeta)\to\eta$, we write the integral J_r (see (A 1)) as follows

$$J_r = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}[3\zeta^2-\xi^2+2\eta\zeta-2\xi(\eta+\zeta)-4\sin^2\theta(\eta+\zeta)(\eta+2\zeta)]\,\mathrm{e}^{-\mathrm{i}\Omega(\zeta+v-\xi)} \times\hat{A}(t_1-v-\zeta)\hat{A}(t_1-v-\zeta-\eta)\hat{A}^*(t_1-v-\zeta-\eta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v + \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}[(\eta+\zeta)^2-\xi^2+2\zeta(\eta+\zeta)-2\xi\zeta-4\sin^2\theta\zeta(\eta+2\zeta)]\,\mathrm{e}^{-\mathrm{i}\Omega(\eta+\zeta+v-\xi)} \times\hat{A}(t_1-v-\zeta)\hat{A}(t_1-v-\zeta-\eta)\hat{A}^*(t_1-v-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v,$$

where additional transformations $\eta \to -\eta$ and $(\zeta+\eta) \to \zeta$ have been made in the second integral so that the integration ranges of both integrals are the same. We also perform the transformation $(\xi-\eta) \to \xi$ to obtain

$$J_r = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} [6\zeta^2+4(\eta-\xi)\zeta-2\xi(2\xi+\eta)-4\sin^2\theta(\eta+2\zeta)^2]\,\mathrm{e}^{-\mathrm{i}\Omega(\zeta-+v-\xi)}$$
$$\times \hat{A}(t_1-v-\zeta)\hat{A}(t_1-v-\zeta-\eta)\hat{A}^*(t_1-v-\zeta-\eta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v$$
$$+\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} [(\eta+\zeta)(\xi+2\eta+3\zeta)-4\sin^2\theta\zeta(\xi+\eta+2\zeta)]\,\mathrm{e}^{-\mathrm{i}\Omega(\zeta+v+\xi)}$$
$$\times \hat{A}(t_1-v-\zeta)\hat{A}(t_1-v-\zeta-\eta-\zeta)\hat{A}^*(t_1-v-\zeta-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v.$$

The second integral can be dropped because it does not contribute to the required jump. After performing the transformation $(v+\zeta)\to\zeta$, and integrating by parts, we can reduce the first quartic integral to a triple integral

$$J_r = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} [2\zeta^3+2(\eta-\xi)\,\zeta^2-2\xi(2\xi+\eta)\,\zeta-4\sin^2\theta(\tfrac{4}{3}\zeta^3+2\eta\zeta^2+\eta^2\zeta]$$
$$\times\,\mathrm{e}^{-\mathrm{i}\Omega(\zeta-\xi)}\hat{A}(\tau_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-\zeta-\eta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

Applying a similar procedure to I_1 and I_2, we finally have that

$$\hat{V}_{3,YY} = \hat{V}^{(r)}_{3,YY}+\hat{V}^{(s)}_{3,YY}+\hat{V}^{(t)}_{3,YY}+\hat{V}^{(b)}_{3,YY}+\dots, \tag{4.84}$$

where
$$\hat{V}^{(s)}_{3,YY} = \bar{S}^4\sin^4\theta I_v = \alpha\bar{S}^{-1}\hat{U}^*_{1,YY}\,\hat{V}^{(2,0)}_{2,Y}, \tag{4.85}$$
$$\hat{V}^{(t)}_{3,YY} = 4\bar{S}^4\sin^6\theta J_v = 2\bar{S}^4\sin^6\theta\hat{W}^{(2)}_0\Pi_0, \tag{4.86}$$
$$\hat{V}^{(b)}_{3,YY} = 2\mathrm{i}\bar{S}^3\sin^2\theta I^{(b)}_v, \tag{4.87}$$
$$\hat{V}^{(r)}_{3,YY} = \bar{S}^4\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\hat{K}_v(\xi,\eta,\zeta)\,\mathrm{e}^{-\mathrm{i}\Omega(\zeta-\xi)}$$
$$\times \hat{A}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-\zeta-\eta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v, \tag{4.88}$$

and

$$\hat{K}_v(\xi,\eta,\zeta) = \tfrac{1}{2}\xi^2(2\xi+\eta)+4\sin^2\theta[\tfrac{1}{2}\zeta^3+\tfrac{1}{2}(\eta-\xi)\zeta^2-\tfrac{1}{2}\xi(\xi+2\eta)\zeta+\tfrac{1}{4}\eta(\xi+\eta)(2\xi+\eta)]$$
$$+4\sin^4\theta[-\zeta^3+(\xi-\eta)\zeta^2+(\xi^2+2\xi\eta)\zeta]. \tag{4.89}$$

The harmonic component $\tilde{V}_3$ satisfies the equation

$$\hat{\mathrm{L}}^{(3)}_0\tilde{V}_{3,YY} = \tilde{M}^{(1)}_Y+\tilde{M}^{(2)}_Y+\tilde{M}^{(3)}_Y. \tag{4.90}$$

The expressions for $\tilde{M}^{(j)}_Y$ $(j=1,2,3)$, as well as the solution for $\tilde{V}_3$ are given in Appendix C.

We now seek the solution for W_3. This term satisfies

$$\mathrm{L}_0 W_3 = -\frac{\partial P_3}{\partial z}-F_2(Y)\frac{\partial W_2}{\partial x}-F_3(Y)\frac{\partial W_1}{\partial x}-S_{32}, \tag{4.91}$$

where
$$F_3(Y) = \tfrac{1}{6}\bar{U}_{yyy}Y^3+\bar{U}_{yy\tau}\tau_1 Y^2+\bar{U}_{y\tau\tau}\tau_1^2 Y+\tfrac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3.$$

We write

$$W_3 = \hat{W}_3 E\sin\beta z+\tilde{W}_3 E^3\sin\beta z+\text{c.c.}+\dots, \tag{4.92}$$

where the components irrelevant to the generation of the two-dimensional fundamental at the next order have not been written out. It is clear from (4.91) that $\hat{W}_3$ should include solutions driven by first three terms on the right-hand side of (4.91). However, these solutions do not contribute to the jump $(C_j^+ - C_j^-)$. So we shall only solve for the solutions driven by $-S_{32}$, i.e. it is sufficient to solve for $\hat{W}_3$ which satisfies

$$\hat{\mathrm{L}}_0^{(1)}\hat{W}_3 = -S_{32}^{(1,1)}. \tag{4.93}$$

Following a similar procedure to that for solving and simplifying $\hat{V}_3$, we have

$$\hat{W}_3 = \hat{W}_3^{(r)} + \hat{W}_3^{(s)} + \hat{W}_3^{(t)} + \hat{W}_3^{(b)}, \tag{4.94}$$

where
$$\hat{W}_3^{(s)} = \tfrac{1}{2}\mathrm{i}\bar{S}^{-1}\hat{W}^*_{1,Y}\hat{V}^{(2,0)}_{2,Y}, \tag{4.95}$$

$$\hat{W}_3^{(t)} = -\mathrm{i}\beta^{-1}\bar{S}^3\sin^6\theta\,\hat{W}_0^{(1)}\Pi_0, \tag{4.96}$$

$$\hat{W}_3^{(b)} = -\beta^{-1}\bar{S}^2\sin^2\theta\, I_w^{(b)}, \tag{4.97}$$

$$\hat{W}_3^{(r)} = -\mathrm{i}\beta^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\hat{K}_w(\xi,\eta,\zeta)\,\mathrm{e}^{-\mathrm{i}\Omega(\zeta-\xi)}$$
$$\times\hat{A}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-\zeta-\eta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{4.98}$$

$$\hat{K}_2(\xi,\eta,\zeta) = -\tfrac{1}{4}\xi^2+\sin^2\theta[\zeta^2+\eta\zeta-\tfrac{1}{2}\eta(\eta+3\xi)]+\sin^4\theta[-(2\xi+\eta)\zeta], \tag{4.99}$$

and the function $I_w^{(b)}$ is given by

$$I_w^{(b)} = \int_0^{+\infty}\int_0^{+\infty}\xi\hat{A}^*(t_1-\xi-\eta)\hat{B}(t_1-\eta)\,\mathrm{e}^{-\mathrm{i}\Omega(\eta-\xi)}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{4.100}$$

The function $\tilde{W}_3$ satisfies the equation

$$\hat{\mathrm{L}}_0^{(3)}\tilde{W}_3 = -S_{32}^{(3,1)}. \tag{4.101}$$

After simplification, the solution can be written

$$\tilde{W}_3 = \tilde{W}_3^{(r)} + \tilde{W}_3^{(s)} + \tilde{W}_3^{(b)}, \tag{4.102}$$

where
$$\tilde{W}_3^{(s)} = \tfrac{1}{2}\mathrm{i}\bar{S}^{-1}\hat{W}_{1,Y}\hat{V}^{(2,0)}_{2,Y},\quad \tilde{W}_3^{(b)} = -\beta^{-1}\bar{S}^2\sin^2\theta\,\tilde{I}_w^{(b)},$$

$$\tilde{W}_3^{(r)} = -\mathrm{i}\beta^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\tilde{K}_w(\xi,\eta,\zeta)\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\zeta+3\eta)}$$
$$\times\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{4.103}$$

$$\tilde{K}_w(\xi,\eta,\zeta) = \tfrac{1}{4}\xi(\xi+2\zeta)+\sin^2\theta(\xi+\zeta)(\eta-\zeta)-\sin^4\theta[3\eta^2+2(\xi+2\zeta)\eta], \tag{4.104}$$

and the function $\tilde{I}_w^{(b)}$ is defined by

$$\tilde{I}_w^{(b)} = \int_0^{+\infty}\int_0^{+\infty}\xi\hat{A}(t_1-\xi-\eta)\hat{B}(t_1-\eta)\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+3\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{4.105}$$

The streamwise velocity at $O(\epsilon^{\frac{4}{3}})$) takes the form

$$U_3 = \hat{U}_3 E\cos\beta z + \tilde{U}_3 E^3\cos\beta z + \text{c.c.} + \dots. \tag{4.106}$$

From the continuity equation it follows that

$$\hat{U}_3 = -(\mathrm{i}\alpha)^{-1}(\hat{V}_{3,Y}+\beta\hat{W}_3), \tag{4.107}$$

$$\tilde{U}_3 = -(3\mathrm{i}\alpha)^{-1}(\tilde{V}_{3,Y}+\beta\tilde{W}_3). \tag{4.108}$$

For the purpose of deriving the amplitude equation, it is sufficient to seek only the two-dimensional fundamental component in V_4, i.e.

$$V_4 = \hat{V}_4^{(2,0)}E^2 + \text{c.c.} + \dots . \tag{4.109}$$

The function $\hat{V}_4^{(2,0)}$ satisfies the following equation:

$$\hat{\mathrm{L}}_0^{(2)}\hat{V}_{4,YY}^{(2,0)} = \hat{\mathrm{L}}_1^{(2)}\hat{V}_3^{(2,0)} + \hat{\mathrm{L}}_2^{(2)}\hat{V}_2^{(2,0)} + 2\mathrm{i}\alpha\,\partial S_{13}^{(2,0)}/\partial Y + 4\alpha^2 S_{21}^{(2,0)}, \tag{4.110}$$

where $S_{13}^{(2,0)}$ and $S_{21}^{(2,0)}$ are the components proportional to E^2 in the Reynolds stresses S_{13} and S_{21} respectively, i.e.

$$S_{13} = \frac{\partial}{\partial x}(2U_1U_3 + U_2^2) + \frac{\partial}{\partial Y}(U_1V_3 + V_1U_3 + U_2V_2) + \frac{\partial}{\partial z}(U_1W_3 + W_1U_3 + U_2W_2),$$

$$S_{21} = (\partial/\partial x)V_1U_1 + (\partial/\partial Y)V_1^2 + (\partial/\partial z)V_1W_1,$$

and

$$S_{13} = S_{13}^{(2,0)}E^2 + \text{c.c.} + \dots, \quad S_{21} = S_{21}^{(2,0)}E^2 + \text{c.c.} + \dots .$$

It is found that the solution driven by $S_{21}^{(2,0)}$ does not contribute to the jump $(C_j^+ - C_j^-)$, and therefore $S_{21}^{(2,0)}$ will not be considered further.

We denote the solution driven by $(\hat{\mathrm{L}}_1^{(2)}\hat{V}_3^{(2,0)} + \hat{\mathrm{L}}_2^{(2)}\hat{V}_2^{(2,0)})$ by $\hat{V}_4^{(l)}E$. Following a similar procedure to that for the two-dimensional case (Wu 1991), we find

$$\hat{V}_{4,Y}^{(l)}(+\infty) - \hat{V}_{4,Y}^{(l)}(-\infty) = \pi\mathrm{i}\,\mathrm{sgn}\,(\bar{U}_y)(a_j^+R_j + p_jD_j^+ + b_jS_j). \tag{4.111}$$

The nonlinear driving term $2\mathrm{i}\alpha\,\partial S_{13}^{(2,0)}/\partial Y$ is complicated, but after some calculation we have

$$\begin{aligned}2\mathrm{i}\alpha\,\partial S_{13}^{(2,0)}/\partial Y = & -2\mathrm{i}\alpha\hat{U}_{1,Y}\hat{V}_{3,Y} + \mathrm{i}\alpha\hat{U}_{1,YY}\hat{V}_3 - 4\mathrm{i}\alpha\beta\hat{U}_{1,Y}\hat{W}_3 \\ & -4\mathrm{i}\alpha\beta\hat{U}_1\hat{W}_{3,Y} - 3\mathrm{i}\alpha\hat{U}_1\hat{V}_{3,YY} - \hat{A}(\hat{V}_{3,YYY} + \beta\hat{W}_{3,YY}) \\ & + \tfrac{2}{3}\mathrm{i}\alpha\hat{U}_{1,Y}^*\tilde{V}_{3,Y} + \mathrm{i}\alpha\hat{U}_{1,YY}^*\tilde{V}_3 - \tfrac{4}{3}\mathrm{i}\alpha\beta\hat{U}_{1,Y}^*\tilde{W}_3 \\ & -\tfrac{4}{3}\mathrm{i}\alpha\beta\hat{U}_1^*\tilde{W}_{3,Y} - \tfrac{1}{3}\mathrm{i}\alpha\hat{U}_1^*\tilde{V}_{3,YY} + 2\mathrm{i}\alpha\hat{U}_{2,YY}^{*(0,0)}\hat{V}_2^{(2,0)} \\ & + [-4\mathrm{i}\alpha\beta\hat{U}_2^{*(0,2)}\hat{W}_{2,Y}^{(2,2)} - \beta\hat{V}_2^{*(0,2)}\hat{W}_{2,YY}^{(2,2)}] \\ & + [-2\mathrm{i}\alpha\hat{U}_2^{*(0,0)}\hat{V}_{2,YY}^{(2,2)} - 4\mathrm{i}\alpha\beta\hat{U}_{2,Y}^{*(0,2)}\hat{W}_2^{(2,2)} \\ & -2\beta\hat{V}_{2,Y}^{*(0,2)}\hat{W}_{2,Y}^{(2,2)} - \beta\hat{V}_{2,YY}^{*(0,2)}\hat{W}_2^{(2,2)}] + \dots,\end{aligned} \tag{4.112}$$

where the dots are to remind the reader that we have ignored forcing terms that do not contribute to the jump.

As already remarked, the $\hat{V}_{3,YY}^{(s)}$ and $\hat{V}_{3,YY}^{(t)}$ terms in $\hat{V}_3$, and the $\hat{W}_3^{(s)}$ and $\hat{W}_3^{(t)}$ terms in $\hat{W}_3$, etc., make the calculation of the asymptotic form of the forced solutions difficult. To overcome this difficulty, we notice that from the expansions of the x-momentum and continuity equations

$$\hat{V}_3 = (\mathrm{i}\alpha\bar{U}_y)^{-1}\hat{\mathrm{L}}_0^{(1)}(\hat{V}_{3,Y} + \beta\hat{W}_3) - \bar{U}_y^{-1}S_{1,2}^{(1,1)}, \tag{4.113}$$

$$\tilde{V}_3 = (3\mathrm{i}\alpha\bar{U}_y)^{-1}\hat{\mathrm{L}}_0^{(3)}(\tilde{V}_{3,Y} + \beta\tilde{W}_3) - \bar{U}_y^{-1}S_{1,2}^{(3,1)}. \tag{4.114}$$

Using the above relations and (4.35), we split $2\mathrm{i}\alpha S_{13,Y}^{(2,0)}$ into four parts, i.e.

$$2\mathrm{i}\alpha S_{13,Y}^{(2,0)} = N^{(r)} + N^{(s)} + N^{(t)} + N^{(b)}. \tag{4.115}$$

The expressions for $N^{(r)}$, etc. are given in Appendix D.†

† Appendix D is available on request from the author or the Editorial Office.

We denote the solutions driven by $N^{(r)}$, $N^{(s)}$, $N^{(t)}$ and $N^{(b)}$ by $\hat{V}_4^{(r)}$, $\hat{V}_4^{(s)}$, $\hat{V}_4^{(t)}$ and $\hat{V}_4^{(b)}$ respectively; thus

$$\hat{V}_4 = \hat{V}_4^{(l)} + \hat{V}_4^{(r)} + \hat{V}_4^{(s)} + \hat{V}_4^{(t)} + \hat{V}_4^{(b)}. \tag{4.116}$$

After integration by parts, we find that

$$\hat{V}^{(s)}_{4,YY} = \tfrac{1}{3}\bar{U}_y^{-1}\hat{U}^*_{1,YY}[\tilde{V}_{3,Y} + \beta\tilde{W}_3] + \hat{U}_y^{-1}\hat{U}_{1,YY}[\hat{V}_{3,Y} + \beta\hat{W}_3] - \tfrac{1}{2}\hat{U}_y^{-2}(\hat{U}_{1,Y}\hat{U}^*_{1,Y})_Y\hat{V}^{(2,0)}_{2,Y} + \bar{U}_y^{-1}\hat{U}^{*(0,0)}_{2,YY}\hat{V}^{(2,0)}_{2,Y}. \tag{4.117}$$

After further integration by parts, we conclude that

$$\hat{V}^{(s)}_{4,Y}(+\infty) - \hat{V}^{(s)}_{4,Y}(-\infty) = 2\mathrm{i}\bar{S}^5\sin^4 j_0\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}K_s(\xi,\eta,\zeta)\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta)$$
$$\times\hat{A}(t_1-\eta-\zeta-\xi)\hat{A}^*(t_1-3\eta-2\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$
$$+2\bar{S}^4\sin^4\theta j_0\int_0^{+\infty}\int_0^{+\infty}\{2\zeta(\eta+\zeta)(\eta+3\zeta)\hat{B}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-3\zeta-\eta)$$
$$+\zeta(\eta+2\zeta)(2\eta+3\zeta)\hat{B}(t_1-\zeta-\eta)\hat{A}(t_1-\zeta)\hat{A}^*(t_1-3\zeta-2\eta)\}\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{4.118}$$

where

$$K_s(\xi,\eta,\zeta) = \tfrac{21}{2}\eta^4 + (3\xi+17\zeta)\eta^3 + (-\tfrac{47}{8}\xi^2 - \tfrac{41}{8}\xi\zeta + \tfrac{113}{16}\zeta^2)\eta^2 + (-\tfrac{13}{4}\xi^3 - \tfrac{23}{2}\xi^2\zeta - \tfrac{45}{4}\xi\zeta^2 - \tfrac{7}{4}\zeta^3)\eta + (-\tfrac{1}{2}\xi^4 - \tfrac{21}{8}\xi^3\zeta - \tfrac{45}{8}\xi^2\zeta^2 - 5\xi\zeta^3 - \tfrac{3}{2}\zeta^4). \tag{4.119}$$

After some manipulation, $\hat{V}^{(t)}_{4,YY}$ can be written as

$$\hat{V}^{(t)}_{4,YY} = \mathrm{i}\bar{S}^5\sin^4\theta[J_t^{(0)} + \sin^2\theta(J_t^{(1)} + J_t^{(2)}) - \sin^4\theta J_t^{(3)}], \tag{4.120}$$

where

$$J_t^{(1)} = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\eta\zeta^2(\eta+2\zeta+2\upsilon)^2\xi^{-1}(\mathrm{e}^{-\mathrm{i}\Omega(\eta+2\zeta+2\upsilon-\xi)} - \mathrm{e}^{-\mathrm{i}\bar{\Omega}(\eta+2\zeta+2\upsilon)})\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(\eta+2\zeta+2\upsilon-\xi)}$$
$$\times\hat{A}(t_1-\zeta-\upsilon)\hat{A}(t_1-\zeta)\hat{A}(t_1-\eta-\zeta-\upsilon)\hat{A}^*(t_1-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}\upsilon, \tag{4.121}$$

$$J_t^{(2)} = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\eta(\zeta+\upsilon)^2(\eta+2\upsilon)^2\xi^{-1}(\mathrm{e}^{-\mathrm{i}\bar{\Omega}(\eta+2\upsilon-\xi)} - \mathrm{e}^{-\mathrm{i}\bar{\Omega}(\eta+2\upsilon)})\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(\eta+2\upsilon-\xi)}$$
$$\times\hat{A}(t_1-\upsilon)\hat{A}(t_1-\zeta-\upsilon)\hat{A}(t_1-\eta-\upsilon)\hat{A}^*(t_1-\zeta-\upsilon-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}\upsilon, \tag{4.122}$$

$$J_t^{(3)} = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\eta\zeta^2(2\eta+\upsilon)(\eta+\upsilon)\xi^{-1}(\mathrm{e}^{-\mathrm{i}\bar{\Omega}(2\eta+\upsilon-\xi)} - \mathrm{e}^{-\mathrm{i}\bar{\Omega}(2\eta+\upsilon)})\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(2\eta+\upsilon)}$$
$$\times\hat{A}(t_1-\zeta)\hat{A}(t_1-\eta)A(t_1-\eta-\upsilon)\hat{A}^*(t_1-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}\upsilon, \tag{4.123}$$

$$J_t^{(0)} = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}K_t^{(0)}(\xi,\eta,\zeta,\upsilon)\,\mathrm{e}^{-\mathrm{i}\Omega(\zeta+2\upsilon+2\mu-\xi)}\hat{A}(t_1-\mu-\eta)$$
$$\times\hat{A}(t_1-\mu-\upsilon)\hat{A}(t_1-\mu-\upsilon-\zeta)\hat{A}^*(t_1-\mu-\eta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}\upsilon\,\mathrm{d}\mu, \tag{4.124}$$

and

$$K_t^{(0)}(\xi,\eta,\zeta,\upsilon) = \tfrac{1}{2}\zeta(\zeta+2\upsilon)(2\xi+4\eta+\zeta+2\upsilon) + \sin^2\theta[-\tfrac{3}{4}(\xi+2\eta)(\zeta^2+2\zeta\upsilon+2\upsilon^2) - \tfrac{1}{2}\upsilon(\zeta+\upsilon)(\zeta+2\upsilon)]. \tag{4.125}$$

The factors $(\eta+2\zeta+2\upsilon)$ in $J_t^{(1)}$, and $(\eta+2\upsilon)$ in $J_t^{(2)}$ follow from our grouping of the forcing terms. The appearance of these factors makes the contributions from different stationary points isolated and therefore resolves the difficulty in obtaining the asymptotic forms of the related solutions.

After some manipulation, it is found that

$$\hat{V}^{(t)}_{4,Y}(+\infty)-\hat{V}^{(t)}_{4,Y}(-\infty) = 2\mathrm{i}\bar{S}^5\sin^4\theta j_0\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} K_t(\xi,\eta,\zeta)\hat{A}(t_1-\eta) \times \hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)\hat{A}^*(t_1-3\eta-2\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \quad (4.126)$$

where

$$K_t(\xi,\eta,\zeta) = -\tfrac{99}{16}\eta^4+\tfrac{11}{4}(\xi-2\zeta)\eta^3+\tfrac{1}{16}(97\xi^2+243\xi\zeta+59\zeta^2)\eta^2 + (\tfrac{15}{8}\xi^3+\tfrac{19}{2}\xi^2\zeta+\tfrac{57}{4}\xi\zeta^2+\tfrac{11}{2}\zeta^3)\eta+(\tfrac{3}{2}\xi^2\zeta^2+3\xi\zeta^3+\tfrac{3}{2}\zeta^4). \quad (4.127)$$

The solution for $\hat{V}^{(b)}_{4,YY}$, and the asymptote of $\hat{V}^{(b)}_{4,Y}$ as $Y\to+\infty$, are straightforward to calculate; the final result is

$$\hat{V}^{(b)}_{4,Y}(+\infty)-\hat{V}^{(b)}_{4,Y}(-\infty) = 2\bar{S}^4\sin^2\theta j_0\int_0^{+\infty}\int_0^{+\infty}\{\{8\zeta^3+\sin^2\theta[-2\zeta(\eta^2+4\eta\zeta+7\zeta^2)]\} \hat{B}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-3\zeta-\eta)+\{4\zeta(\eta+\zeta)(\eta+2\zeta) -\sin^2\theta\zeta(\eta+2\zeta)(6\eta+7\zeta)\}\hat{B}(t_1-\zeta-\eta)\hat{A}(t_1-\zeta)\hat{A}^*(t_1-3\zeta-2\eta)\}\,\mathrm{d}\eta\,\mathrm{d}\zeta. \quad (4.128)$$

The solution driven by $N^{(r)}$ can be readily written out and there is no essential difficulty in obtaining the asymptotes of $\hat{V}^{(r)}_{4,Y}$ as $Y\to\pm\infty$. However, because the solution consists of many terms, it needs a great deal of algebraic manipulation to simplify it; part of this manipulation was done using the computer program *Mathematica*. The final result is

$$\hat{V}^{(r)}_{4,Y}(+\infty)-\hat{V}^{(r)}_{4,Y}(-\infty) = 2\mathrm{i}\bar{S}^5\sin^2\theta j_0\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} K_r(\xi,\eta,\zeta)\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta) \times\hat{A}(t_1-\eta-\zeta-\xi)\hat{A}^*(t_1-3\eta-2\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \quad (4.129)$$

where the kernel is given by

$$K_r(\xi,\eta,\zeta) = \tfrac{183}{16}\eta^4+(-\tfrac{3}{4}\xi+\tfrac{29}{2}\zeta)\eta^3+(-\tfrac{7}{16}\xi^2-\tfrac{77}{16}\xi\zeta+\tfrac{15}{4}\zeta^2)\eta^2 +(\tfrac{11}{8}\xi^3+\tfrac{9}{4}\xi^2\zeta-\tfrac{5}{4}\xi\zeta^2-\tfrac{3}{4}\zeta^3)\eta+(\tfrac{1}{2}\xi^4+\tfrac{21}{8}\xi^3\zeta^2+\tfrac{33}{8}\xi^2\zeta^2+2\xi\zeta^3). \quad (4.130)$$

To match $\hat{V}_{4,Y}$ with the appropriate outer expansion, we require

$$C_j^+-C_j^- = \hat{V}_{4,Y}(+\infty)-\hat{V}_{4,Y}(-\infty). \quad (4.131)$$

Using (4.116), (4.111), (4.118), (4.126), (4.128) and (4.129), we have that

$$C_j^+-C_j^- = \pi\,\mathrm{i}\,\mathrm{sgn}\,(\bar{U}_y)\,(a_j^+R_j+p_jD_j^++b_jS_j) +3\bar{S}^4j_0\int_0^{+\infty}\int_0^{+\infty}\zeta^3\hat{B}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-3\zeta-\eta)\,\mathrm{d}\eta\,\mathrm{d}\zeta +\tfrac{3}{2}\bar{S}^4j_0\int_0^{+\infty}\int_0^{+\infty}\zeta(\eta+\zeta)(\eta+2\zeta)\hat{B}(t_1-\zeta-\eta)\hat{A}(t_1-\zeta)\hat{A}^*(t_1-3\zeta-2\eta)\,\mathrm{d}\eta\,\mathrm{d}\zeta +\tfrac{1}{8}\mathrm{i}\bar{S}^5\sin^2\theta j_0\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}K_b(\xi,\eta,\zeta)\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta) \times\hat{A}(t_1-\eta-\zeta-\xi)\hat{A}^*(t_1-3\eta-2\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \quad (4.132)$$

where the kernel $K_b(\xi,\eta,\zeta)$ is

$$K_b(\xi,\eta,\zeta) \equiv K_r(\xi,\eta,\zeta)+K_s(\xi,\eta,\zeta)+K_t(\xi,\eta,\zeta)$$
$$= -3\eta^4-4(\xi+2\zeta)\,\eta^3-(3\xi^2+\xi\zeta+2\zeta^2)\,\eta^2+(3\xi^2+5\xi\zeta+4\zeta^2)\,\zeta\eta. \quad (4.133)$$

5. Evolution equations for the amplitudes

5.1. *Coupled amplitude equations*

Substituting the jumps (4.24), (4.25) and (4.79) into (3.38), we obtain the following amplitude equation for the oblique waves:

$$\frac{dA}{dt_1} = g_0\,\tau_1 A+g_{11}\int_0^{+\infty}\xi^2 A^*(t_1-2\xi)\,B(t_1-\xi)\,d\xi$$
$$+g_{12}\int_0^{+\infty}\int_0^{+\infty}K_a(\xi,\eta)\,A(t_1-\xi)\,A(t_1-\xi-\eta)\,A^*(t_1-2\xi-\eta)\,d\xi\,d\eta, \quad (5.1)$$

where we have written $g_0=f_0/f$, and $g_{1k}=f_{1k}/f\,(k=1,2)$. The constants f and f_0 have the same expressions as for a single two-dimensional wave (Wu 1991), i.e.

$$f = i\alpha^{-1}\left\{\sum_j \pi b_j\left[2i\frac{\bar{U}_{yy}}{|\bar{U}_y|^2}a_j^+ + ib_j\frac{\bar{U}_y\bar{U}_{yyy}-\bar{U}_{yy}^2}{\bar{U}_y\bar{U}_y^2}+b_j\pi\frac{\bar{U}_{yy}^2}{\bar{U}_y^3}\right]+J_1\right\},$$

$$f_0 = \sum_j\left\{2\pi i b_j\frac{\bar{U}_{yy}\bar{U}_\tau}{\bar{U}_y^2}a_j^+ - \pi i b_j^2\left[\frac{\bar{U}_{yy\tau}}{|\bar{U}_y|}-\frac{\bar{U}_{yy}\bar{U}_{y\tau}}{\bar{U}_y^2}-\frac{(\bar{U}_y\bar{U}_{yyy}-\bar{U}_{yy}^2)\bar{U}_\tau}{\bar{U}_y^3}\right]+\pi^2 b_j^2\frac{\bar{U}_{yy}\bar{U}_\tau}{\bar{U}_y^3}\right\}-J_2,$$

where the sum is over all critical layers (there are two in the Stokes layer under consideration in this paper), and J_1 and J_2 are defined by (3.39) and (3.40) respectively. The constants f_{11} and f_{12} are

$$f_{11} = -3\pi i\alpha^2\sum_j b_j|b_j|^2|\bar{U}_y|^2, \quad (5.2)$$

$$f_{12} = \tfrac{3}{16}\pi\alpha^3\sum_j b_j^2|b_j|^2|\bar{U}_y|^3. \quad (5.3)$$

The kernel $K_a(\xi,\eta)$ is given by

$$K_a(\xi,\eta) = \xi(4\xi^2+5\xi\eta+3\eta^2). \quad (5.4)$$

Similarly the amplitude equation for the two-dimensional wave is

$$\frac{dB}{dt_1} = 2g_0\,\tau_1 B+g_{21}\int_0^{+\infty}\int_0^{+\infty}\zeta^3 B(t_1-\zeta)\,A(t_1-\zeta-\eta)\,A^*(t_1-3\zeta-\eta)\,d\eta\,d\zeta$$
$$+g_{22}\int_0^{+\infty}\int_0^{+\infty}\zeta(\eta+\zeta)(\eta+2\zeta)\,B(t_1-\zeta-\eta)\,A(t_1-\zeta)\,A^*(t_1-3\zeta-2\eta)\,d\eta\,d\zeta$$
$$+g_{23}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}K_b(\xi,\eta,\zeta)\,A(t_1-\eta)\,A(t_1-\eta-\zeta)$$
$$\times A(t_1-\eta-\zeta-\xi)\,A^*(t_1-3\eta-2\zeta-\xi)\,d\xi\,d\eta\,d\zeta, \quad (5.5)$$

where we have written $g_{2k} = 2f_{2k}/f\,(k = 1,2,3)$, the kernel $K_b(\xi, \eta, \zeta)$ is defined by (4.133), and

$$f_{21} = -3\pi\alpha^3 \sum_j b_j^2 |b_j|^2 |\bar{U}_y|^3, \tag{5.6}$$

$$f_{22} = -\tfrac{3}{2}\pi\alpha^3 \sum_j b_j^2 |b_j|^2 |\bar{U}_y|^3, \tag{5.7}$$

$$f_{23} = -\tfrac{3}{32}\alpha^4 \pi \mathrm{i} \sum_j b_j^3 |b_j|^2 |\bar{U}_y|^4. \tag{5.8}$$

Essentially the same amplitude equations have been obtained by Goldstein & Lee (1992) in their study of resonant-triad waves in adverse-pressure-gradient boundary layers. This is an interesting result, considering the apparent difference between the two flows and the fact that the Rayleigh waves in their study have long wavelengths, while in ours the modes have order-one wavelengths. The coefficients of the amplitude equations in the two studies are somewhat different however. Owing to their long-wavelength limit, Goldstein & Lee (1992) were able to give simple expressions for the coefficients. In our study, the expressions for the coefficients are more complicated, involving the basic-flow profile and the eigenfunction of the Rayleigh equation. We believe that the expressions given in this paper are applicable to many other flows, including flows where the critical layers occur at inflexion points. Note that in Goldstein & Lee (1992), the coefficients of nonlinear terms are purely imaginary. This is a consequence of the long-wavelimit approximation and the fact that only one critical layer exists in their flow. In our formation there can exist more than one critical layer with the result that the coefficients can be complex. In addition, the linear growth rate of our oblique waves is half that of the planar wave, rather than the four-fifths obtained by Goldstein & Lee (1992) for the case of long waves.

Note that the first nonlinear term on the right-hand side of (5.1) represents the resonant interaction between the two-dimensional and the three-dimensional waves, and the second represents the self-interaction of the three-dimensional waves. The three-dimensional waves affect the development of the two-dimensional wave through the mutual cubic interaction with the two-dimensional wave, as well as through the quartic self-interaction of the three-dimensional waves. This is very different from conventional resonant-triad amplitude equations, where the three-dimensional waves affect the two-dimensional wave through a quadratic interaction. As pointed out by Goldstein & Lee (1992), the absence of quadratic feedback from the three-dimensional waves is because the two-dimensional wave component generated by the quadratic interaction between oblique waves causes no jump across the critical layers and therefore does not appear in the amplitude equation. We further note that this conclusion holds even when viscosity is retained at the leading order of the critical-layer equations.

In order to evaluate the coefficients in (5.1) and (5.5), we need to solve the Rayleigh equation. This is done for the Stokes layer by the same method as described in Wu (1991). As there we concentrate on neutral modes on the right-hand branch of curve A, although the analysis is equally valid near the left-hand branch (see §2). The plot of $(2g_0)$ against the wavenumber is the same as in figure 3 of Wu (1991) and Wu & Cowley (1992), provided α is replaced by 2α; the real part of g_0 is always negative. The coefficients g_{11}, g_{21}, and g_{23} are plotted against the wavenumber α in figures 3(*a*) and 3(*b*), while g_{12} and g_{22} are related to g_{21} by $g_{12} = -\frac{1}{16} g_{21}$, and $g_{22} = \frac{1}{2} g_{21}$.

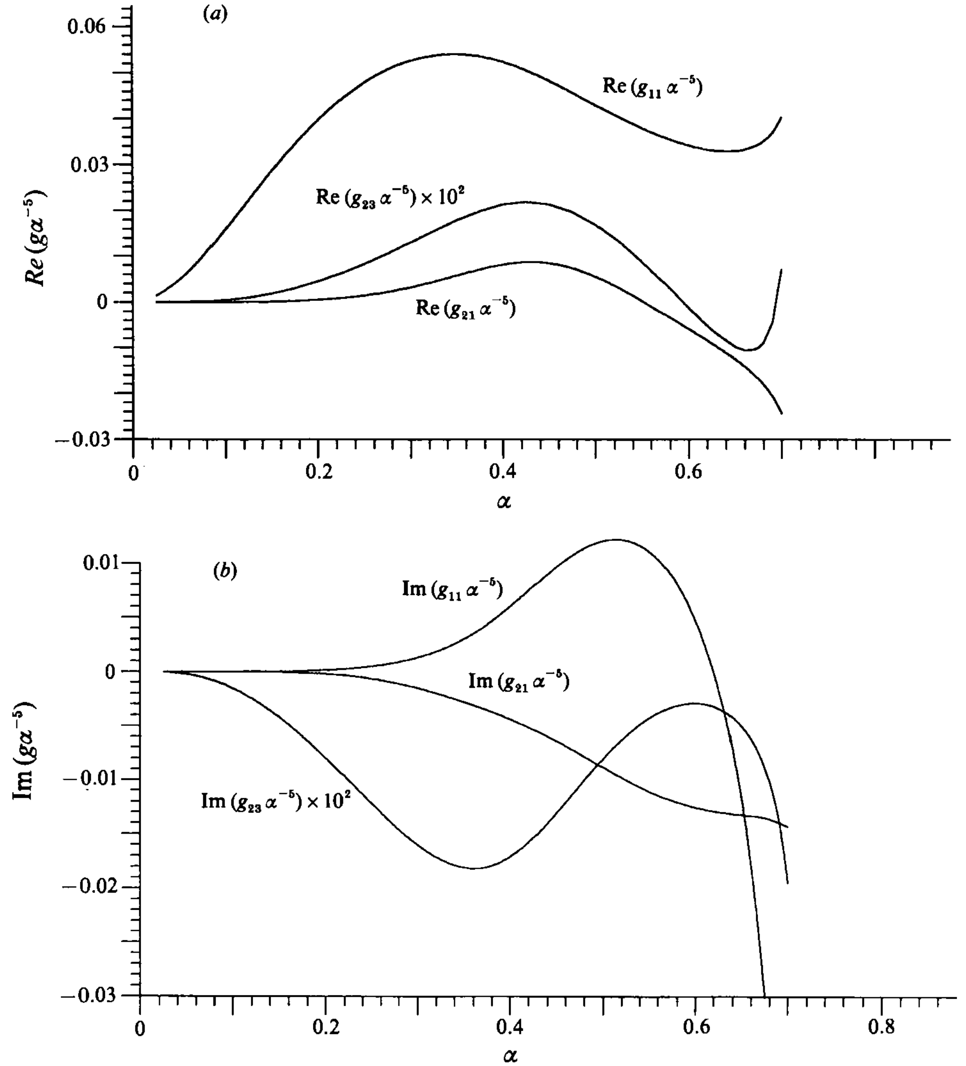

FIGURE 3. (*a*) The real parts and (*b*) the imaginary parts of the coefficients g_{11}, g_{21} and g_{23} in the coupled amplitude equations (5.1) and (5.5).

In order to match to the earlier linear stage in an asymptotic sense, the amplitudes A and B should have the following asymptotic behaviour (see e.g. Goldstein & Leib 1989)

$$A \to A_0 e^{g_0 \tau_1 t_1}, \quad B \to B_0 e^{2g_0 \tau_1 t_1} \quad \text{as} \quad t_1 \to -\infty. \tag{5.9}$$

Thus the evolution of the resonant-triad waves is described by the above coupled amplitude equations (5.1) and (5.5) together with the appropriate 'initial' conditions (5.9).

As demonstrated in §2, in order for there to be a feedback mechanism, the magnitude of the oblique waves is required to be larger than that of the planar wave. To illustrate this point more clearly, we rescale the amplitude equations by introducing the following variables

$$\bar{t} = -\tau_1 t_1 - t_{10}, \quad \bar{B} = B\, e^{iT_B}/(-\tau_1)^4, \quad \bar{A} = A\, e^{iT_A}/\lambda,$$

where the real constants t_{10}, T_A, T_B and λ are chosen so that

$$e^{-iT_B}(-\tau_1)^4 = B_0\,e^{-2g_0 t_{10}}, \quad e^{-iT_A}\lambda = A_0\,e^{-g_0 t_{10}}.$$

The rescaled amplitude equations and asymptotic conditions are as follows:

$$\frac{d\bar{A}}{d\bar{t}} = -g_0\bar{A} + g_{11}\,e^{-i\phi_0}\int_0^{+\infty}\xi^2\bar{A}^*(\bar{t}-2\xi)\,\bar{B}(\bar{t}-\xi)\,d\xi$$
$$+\lambda_0 g_{12}\int_0^{+\infty}\int_0^{+\infty}K_a(\xi,\eta)\,\bar{A}(\bar{t}-\xi)\,\bar{A}(\bar{t}-\xi-\eta)\,\bar{A}^*(\bar{t}-2\xi-\eta)\,d\xi\,d\eta, \quad (5.10)$$

$$\frac{d\bar{B}}{dt} = -2g_0\bar{B} + \lambda_0 g_{21}\int_0^{+\infty}\int_0^{+\infty}\zeta^3\bar{B}(\bar{t}-\zeta)\,\bar{A}(\bar{t}-\zeta-\eta)\,\bar{A}^*(\bar{t}-3\zeta-\eta)\,d\eta\,d\zeta$$
$$+\lambda_0 g_{22}\int_0^{+\infty}\int_0^{+\infty}\zeta(\eta+\zeta)(\eta+2\zeta)\,\bar{B}(\bar{t}-\zeta-\eta)\,\bar{A}(\bar{t}-\zeta)\,\bar{A}^*(\bar{t}-3\zeta-2\eta)\,d\eta\,d\zeta$$
$$+\lambda_0^2\,e^{i\phi_0}g_{23}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}K_b(\xi,\eta,\zeta)\,\bar{A}(\bar{t}-\eta)\,\bar{A}(\bar{t}-\eta-\zeta)$$
$$\times\bar{A}(\bar{t}-\eta-\zeta-\xi)\,\bar{A}^*(\bar{t}-3\eta-2\zeta-\xi)\,d\xi\,d\eta\,d\zeta, \quad (5.11)$$

$$\bar{A}\rightarrow e^{-g_0\bar{t}}, \quad B\rightarrow e^{-2g_0\bar{t}} \quad \text{as} \quad \bar{t}\rightarrow-\infty, \quad (5.12)$$

where $\lambda_0 = |A_0^2/(B_0\tau_1^2)|$, and $\phi_0 = \arg[A_0^2/(B_0\tau_1^2)]$. The real parameter λ_0 accounts for the effect of the initial amplitude of the three-dimensional waves, while ϕ_0 represents the phase difference between the three- and two-dimensional waves.

Up to now we have assumed that the three-dimensional waves are initially stronger than the two-dimensional wave. If on the other hand, $\epsilon \ll \delta^{\frac{3}{4}}$, i.e. $|\lambda_0| \ll 1$, then these equations reduce to

$$\frac{d\bar{A}}{d\bar{t}} = -g_0\bar{A} + g_{11}\,e^{-i\phi_0}\int_0^{+\infty}\xi^2\bar{A}^*(\bar{t}-2\xi)\,\bar{B}(\bar{t}-\xi)\,d\xi, \quad (5.13)$$

$$\frac{d\bar{B}}{d\bar{t}} = -2g_0\bar{B}. \quad (5.14)$$

These are the reduced equations derived by Goldstein & Lee (1992) assuming that all the waves have an equal amplitude initially. It is clear that the three-dimensional waves have no back reaction on the two-dimensional wave. This is the so-called parametric resonance. We note that it can occur whenever $\delta \geqslant O[(\alpha c_i)^4] \gg \epsilon^{\frac{4}{3}}$ and that its effect is negligible when $\delta \ll (\alpha c_i)^4$. As Goldstein & Lee (1992) show, in the parametric-resonance regime the three-dimensional waves experience a faster-than-exponential growth, while the two-dimensional wave continues to grow exponentially. Ultimately the magnitude of the three-dimensional waves becomes equal to $\delta^{\frac{3}{4}}$, at which point the double and triple integrals in (5.10) and (5.11) become so large that the cubic and quartic terms can no longer be neglected. The evolution then enters the regime described by the fully coupled (5.10) and (5.11). Although an asymptotic matching of the two regimes could be constructed with a careful shift of the time origin, it seems unnecessary since the fully coupled equations keep all the terms of (5.13) and (5.14), and are uniformly valid in the two regimes.

5.2. *Finite-time singularity structure*

It seems that solutions of (5.10) and (5.11) develop a finite-time singularity. The structure for it is given by

$$\bar{A} \sim \frac{a_0}{(t_s-\bar{t})^{3+i\sigma}}, \quad \bar{B} \sim \frac{b_0}{(t_s-\bar{t})^{4+2i\sigma}}, \tag{5.15}$$

as $\bar{t} \rightarrow t_s$, where a_0, b_0 are complex numbers and σ is a real number. In fact the structure depends only on the order of the polynomial kernels in the amplitude equations.

Substitution of the above expressions for $\bar{A}$ and $\bar{B}$ into (5.10) and (5.11) yields

$$(3+i\sigma)a_0 = e^{-i\phi_0}g_{11}D_{11}a_0^* b_0 + \lambda_0 g_{12}D_{12}a_0|a_0|^2, \tag{5.16}$$

$$(4+2i\sigma)b_0 = \lambda_0(g_{21}D_{21}+g_{22}D_{22})b_0|a_0|^2 + \lambda_0^2 e^{i\phi_0}g_{23}D_{23}a_0^2|a_0|^2, \tag{5.17}$$

where the constants D_{11} etc. are given by the following convergent integrals:

$$D_{11} = \int_0^{+\infty} \xi^2(1+\xi)^{-(4+2i\sigma)}(1+2\xi)^{-(3-i\sigma)}\,d\xi,$$

$$D_{12} = \int_0^{+\infty}\int_0^{+\infty} K_a(\xi,\eta)\,[(1+\xi)(1+\xi+\eta)]^{-(3+i\sigma)}(1+2\xi+\eta)^{-(3-i\sigma)}\,d\xi\,d\eta,$$

$$D_{21} = \int_0^{+\infty}\int_0^{+\infty} \zeta^3(1+\zeta)^{-(4+2i\sigma)}(1+\zeta+\eta)^{-(3+i\sigma)}(1+3\zeta+\eta)^{-(3-i\sigma)}\,d\eta\,d\zeta,$$

$$D_{22} = \int_0^{+\infty}\int_0^{+\infty} \zeta(\eta+\zeta)(\eta+2\zeta)(1+\zeta+\eta)^{-(4+2i\sigma)}(1+\zeta)^{-(3+i\sigma)}(1+3\zeta+2\eta)^{-(3-i\sigma)}\,d\eta\,d\zeta,$$

$$D_{23} = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} K_b(\xi,\eta,\zeta)\,(1+3\eta+2\zeta+\xi)^{-(3-i\sigma)} \times [(1+\eta)(1+\eta+\zeta)(1+\eta+\zeta+\xi)]^{-(3+i\sigma)}\,d\xi\,dh\,d\zeta.$$

Solving b_0 from (5.17) and substituting into (5.16), we have

$$3+i\sigma = \frac{\lambda_0^2 g_{11}g_{23}D_{11}D_{23}}{(4+2i\sigma)-\lambda_0(g_{21}D_{21}+g_{22}D_{22})|a_0|^2}|a_0|^4 + \lambda_0 g_{12}D_{12}|a_0|^2. \tag{5.18}$$

The parameters $|a_0|$ and σ can be determined from (5.18). After integrating (5.10) and (5.11) numerically, the singularity time t_s can be determined as described in Wu (1991), i.e. by fitting the numerically calculated functions $\bar{A}$ and $\bar{B}$ to (5.15). Surprisingly, the two-dimensional wave has a 'more singular' structure than the three-dimensional waves, as can be seen from (5.15). This implies that as the singularity is approached the two-dimensional wave will grow faster than the three-dimensional waves – eventually attaining a magnitude comparable with that of the three-dimensional waves. Indeed, when the fully nonlinear Euler stage is reached, the two- and the three-dimensional waves attain an order-one magnitude (see below).

The singularity structure (5.15) was also proposed independently by Goldstein & Lee (1992), and has been confirmed by the numerical solutions of their coupled amplitude equations. As mentioned earlier, the coefficients in their equations differ from those in the present study. Therefore, we integrate our amplitude equations (5.10) and (5.11) numerically using the coefficients calculated for the Stokes layer. The parameters are λ_0, ϕ_0 and wavenumber α. Both predictor-corrector (Milne's

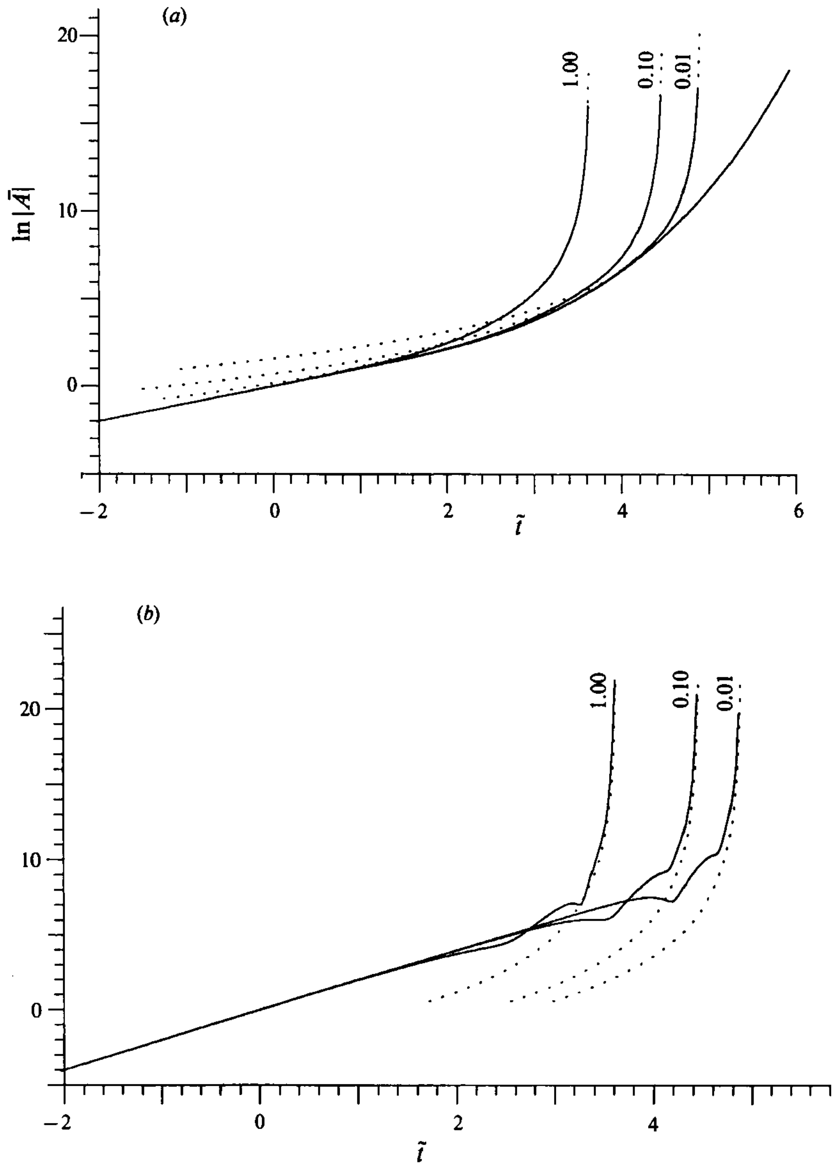

FIGURE 4. (*a*) $\ln|\bar{A}|$, and (*b*) $\ln|\bar{B}|$ *vs.* the scaled time $\tilde{t} = -g_0\bar{t}\,(g_{0r} \equiv \mathrm{Re}\,(g_0) < 0)$ for $\alpha = 0.6$, $\phi_0 = 0$. The parameter $\lambda_0 = 1.0, 0.1, 0.01$ and 0. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (5.15).

method) and Adams–Moulton methods with sixth-order accuracy were employed to check the reliability of the solutions. We present our results in figures 4–6. Figures 4(*a*) and 4(*b*) show the amplitude development of the oblique and planar waves respectively; $\alpha = 0.6$ and $\phi_0 = 0$. Results for $\alpha = 0.4$ and $\phi_0 = 0$ are displayed in figures 5(*a*) and 5(*b*). It is seen that a singularity always seems to occur except when $\lambda_0 = 0$, i.e. except for the parametric resonance case (Goldstein & Lee 1992). As λ_0 increases, i.e. as the initial amplitude of the oblique waves is increased, blow-up occurs earlier. When λ_0 is small, for example $\lambda_0 = 0.01$, the disturbances experience

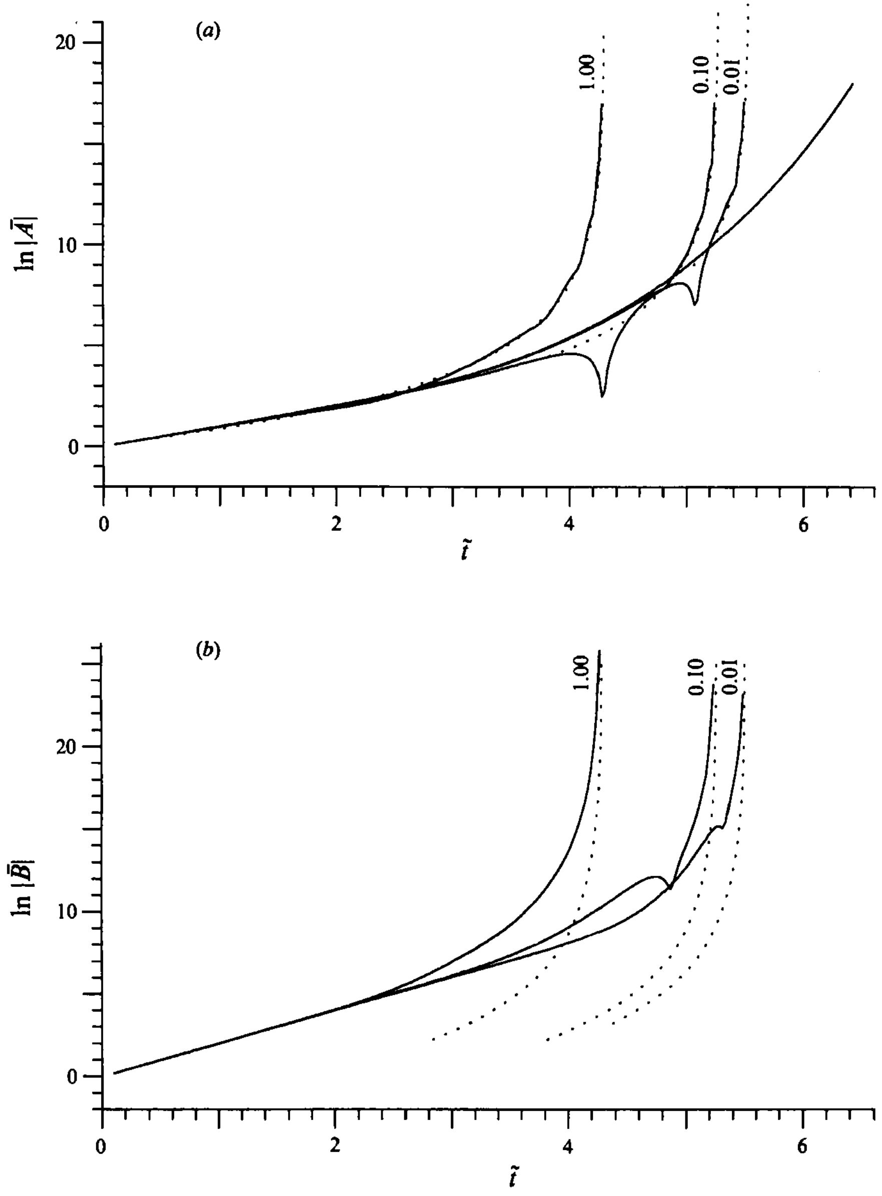

FIGURE 5. (*a*) $\ln|\bar{A}|$ and (*b*) $\ln|\bar{B}|$ *vs.* the scaled time $\tilde{t} = -g_{0r}\bar{t}$ for $\alpha = 0.4$, $\phi_0 = 0$. The parameter $\lambda_0 = 1.0, 0.1, 0.01$ and 0. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (5.15).

a parametric-resonance stage before entering the fully interactive stage. However, when λ_0 is larger, say 1.0, the parametric-resonance stage is not obvious, and the disturbances seem to enter the fully interactive stage directly.

The effect of the phase difference, ϕ_0, on the evolution of the waves is illustrated in figures 6(*a*) and 6(*b*) for $\alpha = 0.4$ and $\lambda_0 = 0.1$. It is seen that in the range of ϕ_0 investigated, i.e. $0 \leqslant \phi_0 \leqslant \frac{3}{4}\pi$, an increase of ϕ_0 delays the occurrence of the singularity; of course this trend is reversed as ϕ_0 is increased further. An important feature is that the instantaneous amplitude of the oblique waves depends on the

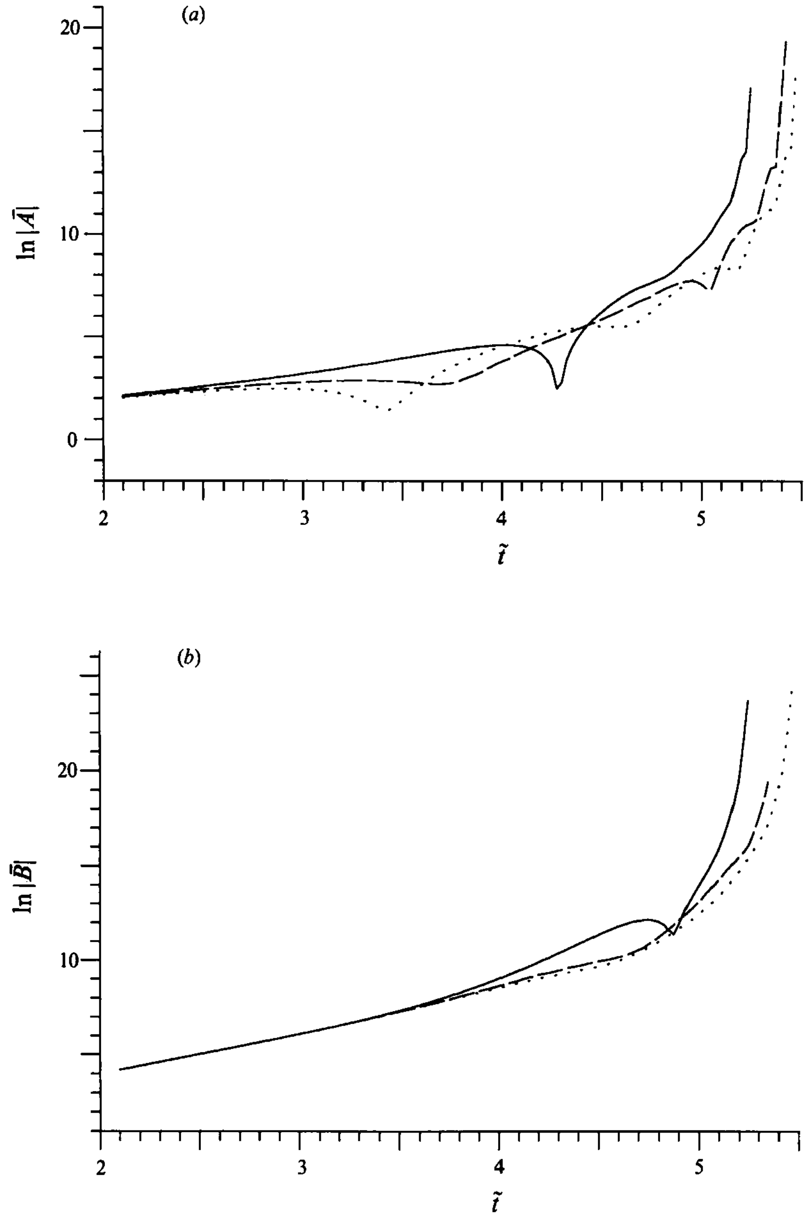

FIGURE 6. (*a*) $\ln|\bar{A}|$ and (*b*) $\ln|\bar{B}|$ *vs*. the scaled time $\tilde{t} = -g_{0r}\bar{t}$ for $\alpha = 0.4$, $\lambda_0 = 0.1$. The parameter $\phi_0 = 0$ (solid line), $\frac{1}{2}\pi$ (dashed line), and $\frac{3}{4}\pi$ (dotted line).

phase difference rather sensitively. This suggests that the phase difference will have to be specified if a meaningful quantitative comparison between theory and experiments is to be made.

Before closing this section, we turn to examine the validity of the amplitude equations (5.10) and (5.11). Following an idea of Goldstein & Leib (1989), we examine the asymptotic growth rates in the vicinity of the singularity. From the singularity structure (5.15), it follows that asymptotic growth rates $|A^\dagger|_t/|A^\dagger|$ and $|B^\dagger|_t/|B^\dagger|$ behave like $(|A^\dagger|/|a_0|)^{\frac{1}{3}}$ and $(|B^\dagger|/|b_0|)^{\frac{1}{4}}$ respectively, where $A^\dagger = \epsilon A$ and $B^\dagger = \delta B$ are the unscaled amplitudes of the three- and two-dimensional waves respectively. Therefore the initial scaling, i.e.

$$\text{growth rate} \sim (A^{\dagger})^{\frac{1}{8}} \sim (B^{\dagger})^{\frac{1}{4}},$$

is unchanged by the singularity. As Goldstein & Leib (1989) observe 'this suggests that the basic asymptotic structure will remain intact, and the present solution will not break down until the amplitude of the disturbance becomes order one. The flow will then be fully nonlinear and unsteady in the whole field, i.e. it will be governed by the Euler equations'. This property of Hickernell type of amplitude equations has been found in other flows, e.g. Goldstein & Leib (1989), Goldstein & Choi (1989).

6. Discussion and conclusions

In this paper, we have discussed a resonant-triad interaction where all three waves interplay and reinforce each other. The resulting amplitude equations are different from those of Raetz (1959), Craik (1971) and Smith & Stewart (1987) in two important aspects. (This comparison is of mathematical interest, and does *not* mean that our analysis applies to the Blasius boundary layer.) Firstly, the local growth rates depend on the whole history of evolution, unlike a conventional resonant triad where the local growth rates depend only on the instantaneous amplitudes of waves. Secondly, the back reaction of the oblique waves on the two-dimensional wave is accounted by two cubic terms and one quartic term, rather than only one quadratic term. The solution of our amplitude equations can always develop a finite-time singularity. The occurrence of a finite-time singularity in the conventional resonant-triad equations is also common, but tends to depend on the coefficients (Craik 1975). For instance, Smith & Stewart (1987) found that no finite-time singularity was allowed in their resonant-triad equations for 'high-frequency' lower-branch Tollmien–Schlichting waves in a Blasius boundary layer. While it would be interesting to see whether our singularity could be removed by viscous effects, to have included them here would have greatly complicated an already complex analysis. In the case of a two-dimensional wave, it has been shown that viscous effects generally delay the occurrence of the finite-time singularity, and in certain cases, sufficiently strong viscosity can eliminate the singularity completely (e.g. Goldstein & Leib 1989). However, Wu (1991) and Wu & Cowley (1992) have shown that for two-dimensional disturbances in a Stokes layer, there is a large range of wavenumbers for which nonlinearity has a strong effect in the sense that a singularity always occurs no matter how large the scaled viscosity parameter is. See also Wu *et al.* (1992) for a study of the viscous effects on disturbances consisting of a pair of oblique waves.

The kernels in the amplitude equations are fully determined by nonlinear interactions inside the critical layers and do not depend on the detailed profile of the basic flow. Therefore, the amplitude equations are universally applicable to any basic flow which supports Rayleigh instability waves, provided appropriate conditions such as those listed at the end of §2 are met. Moreover, we have argued that the amplitude equations remain valid until the amplitude of the disturbance becomes order one; the equations thus provide a full description of the evolution of a resonant-triad of waves from their linear small-amplitude stage up to the fully nonlinear order-one-amplitude stage. The parametric resonance can be treated as a special limit of the fully interactive case.

The observation that an initially small disturbance can be further amplified by nonlinearity provides a possible explanation for the instability and transition of the Stokes layer on a flat plate. It appears that instability and transition may be caused by this nonlinear instantaneous growth rather than by the net growth of disturbances over a period as predicted by Floquet theory. Since the disturbances that we considered have high frequencies, we believe that the bursting phenomenon observed in oscillatory Stokes layers (e.g. Merkli & Thomann 1975) is related to the explosive growth predicted here.

As in Goldstein & Choi (1989), the nonlinear interaction between waves drives a strong mean flow with the same magnitude as the oblique waves (cf. Hall & Smith 1991). The growth of this mean flow is enhanced by the resonance mechanism. Moreover, both the measurements and visualization of Hino *et al.* (1976, 1983) reveal the existence of a vortex structure in unstable Stokes layers.

Besides the above overall qualitative agreement of our theoretical predictions with experimental observations, there are some quantitative conclusions that may be drawn by combining the present nonlinear results with the linear results of Cowley (1987). Firstly, it was shown by Cowley that the largest possible wavenumber for a growing mode was $\alpha_m = 1.43$, and we have shown that for a pair of three-dimensional modes $(\alpha, \pm\beta, c)$, resonance with a two-dimensional mode $(2\alpha, 0, c)$ occurs if $\beta = \surd 3\alpha$. Obviously it is required that $2\alpha \leqslant \alpha_m$, i.e. $\alpha \leqslant 0.72$; then

$$\beta = \surd 3\alpha \leqslant 1.25.$$

Suppose that the disturbances are basically three-dimensional and amplified by resonance, then the streamwise wavelength, say λ_x, of the observed disturbance should be larger than $8.73\delta^*$, i.e.

$$\lambda_x \geqslant 8.73\delta^*,$$

and the spacing, say λ_z, between two neighbouring streaks should be larger than $2.52\delta^*$, i.e.

$$\lambda_z \geqslant 2.52\delta^*,$$

where δ^* is thickness of Stokes layer. Thus our theory gives a 'lower bound' of space scales of observable disturbances. Monkewitz & Bunster (1987) estimated from their experiment that λ_x is about $10\delta^*$. Hino *et al.* (1983) measured λ_z, and found that it was 1.5 cm, which is $3.7\delta^*$. Both seem to be in the predicted range. Secondly our nonlinear theory shows that disturbances are substantially amplified by nonlinear effects in the neighbourhood of neutral curves, which means that initially small disturbances can attain a finite amplitude by this nonlinear amplification. Referring to Cowley's (1987) neutral diagram, we may conclude that very small-amplitude initial disturbances are most likely to be observed shortly before or around the phase $\frac{1}{2}\pi$ of the basic flow. This conclusion is in agreement with the observation of Monkewitz & Bunster (1987). But others, e.g. Akhavan *et al.* (1991*a*) and references herein, showed that explosive growth occurs at the end of the acceleration phase, i.e. phase 0 in the notation of the present study. It is possible that this discrepancy arises because of different levels of background noise in the various experiments (see Wu *et al.* 1992 for a discussion of this point). However, we also note that all the experiments were actually conducted for finite Stokes layers, where an additional parameter $\Lambda = h/\delta^*$ occurs (here h is either the half-width of a channel or the radius of a pipe). The value of Λ in most experiments (including that of Merkli & Thomann 1975) is in the range of 3 to 10, but it is about 44 in the experiment of Monkewitz & Bunster (1987). Though this difference does not seem to alter the basic flow significantly, it is possible that at low Λ, some new mode which does not exist at high Λ may bifurcate and cause transition.

Finally we note that various extensions of our theory are possible. For instance, the oblique waves could be allowed to have different amplitudes, viscous effects could be included in the critical layers, and wavetrains modulated in both the spanwise and streamwise directions could be allowed for.

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Appendix A

In (4.71), we have that

$$J = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi^{-1}[\xi^2+\xi\zeta+2\xi\eta+4\sin^2\theta(\eta+\zeta)\eta]\,[(\zeta-\xi)\,e^{-i\bar{\Omega}(\zeta+v-\xi)} - \zeta e^{-i\bar{\Omega}(\zeta+v)}]\,\hat{A}(t_1-v-\zeta)\,\hat{A}(t_1-v-\eta)\,\hat{A}^*(t_1-v-\eta-\xi)\,e^{-i\bar{\omega}(\zeta+v-\xi)}\,d\xi\,d\eta\,d\zeta\,dv,$$

The factor ξ^{-1} in the kernel leads to difficulties in deriving the asymptotic behaviour of $\int_{-Y}^{+Y} J(Y)\,dY$; this is needed to obtain the jump $(c_j^+-c_j^-)$. To resolve this problem, we split J into three parts, namely J_r, J_v, J_0:

$$J = J_r + 4\sin^2\theta J_v - J_0, \tag{A 1}$$

where

$$J_r = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} [\zeta^2-\xi^2+2\eta\zeta-2\xi\eta-4\sin^2\theta(\eta+\zeta)\eta]\,e^{-i\Omega(\zeta+v-\xi)} \times\hat{A}(t_1-v-\zeta)\,\hat{A}(t_1-v-\eta)\,\hat{A}^*(t_1-v-\eta-\xi)\,d\xi\,d\eta\,d\zeta\,dv,$$

$$J_v = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi^{-1}(\eta+\zeta)\,\eta\zeta[e^{i\bar{\Omega}(\zeta+v-\xi)} - e^{-i\bar{\Omega}(\zeta+v)}] \times\hat{A}(t_1-v-\zeta)\,\hat{A}(t_1-v-\eta)\,\hat{A}^*(t_1-v-\eta-\xi)\,e^{-i\bar{\omega}(\zeta+v-\xi)}\,d\xi\,d\eta\,d\zeta\,dv,$$

$$J_0 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} [\xi+2\eta+\zeta]\,\zeta e^{-i\Omega(\zeta+v)} \times\hat{A}(t_1-v-\zeta)\,\hat{A}(t_1-v-\eta)\,\hat{A}^*(t_1-v-\eta-\xi)\,e^{i\bar{\omega}\xi}\,d\xi\,d\eta\,d\zeta\,dv.$$

In fact the integral J_0 turns out not to contribute to the jumps that we require, and thus will not be considered henceforth.

Integration of J_r with respect to Y from $-\infty$ to $+\infty$ yields

$$\int_{-\infty}^{+\infty} J_r(Y)\,dY = 2j_0\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} [\zeta^2-(\zeta+v)^2+2\eta\zeta-2(\zeta+v)\eta-4\sin^2\theta(\eta+\zeta)\eta] \times\hat{A}(t_1-v-\eta)\,\hat{A}(t_1-v-\zeta)\,\hat{A}^*(t_1-2v-\eta-\zeta)\,d\eta\,d\zeta\,dv.$$

After simplification of the right-hand side, the above equation can be written

$$\int_{-\infty}^{+\infty} J_r(Y)\,\mathrm{d}Y = -2j_0 \int_0^{+\infty}\int_0^{+\infty} [2\xi^2(\xi+\eta)+4\sin^2\theta(\tfrac{4}{3}\xi^3+2\xi^2\eta+\eta^2\xi)] \times \hat{A}(t_1-\xi)\,\hat{A}(t_1-\xi-\eta)\,\hat{A}^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \quad (\mathrm{A}\ 2)$$

The integral J_v can be reduced to a triple-integral form

$$J_v = \tfrac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi^{-1}\eta^2\zeta^2[\mathrm{e}^{-\mathrm{i}\bar{\Omega}(\eta-\xi)}-\mathrm{e}^{-\mathrm{i}\bar{\Omega}\eta}] \times \hat{A}(t_1-\zeta)\,\hat{A}(t_1-\eta)\,\hat{A}^*(t_1-\zeta-\xi)\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(\eta-\xi)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \quad (\mathrm{A}\ 3)$$

or the alternative form

$$J_v = \tfrac{1}{2}\hat{W}_0^{(2)}\Pi_0, \quad (\mathrm{A}\ 4)$$

where

$$\Pi_0 = \int_0^{+\infty}\int_0^{+\infty} \xi^{-1}\zeta^2[\mathrm{e}^{\mathrm{i}\bar{\Omega}\xi}-1]\,\mathrm{e}^{\mathrm{i}\bar{\omega}\xi}\hat{A}(t_1-\zeta)\,\hat{A}^*(t_1-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\zeta. \quad (\mathrm{A}\ 5)$$

It will be seen that the factor η in the kernel of J_v (see (A 3)), which has been introduced by combining two Reynolds stress terms, is important for the success of the following procedure for analysing the asymptote of J_v. Integrating J_v with respect to Y, we have that

$$\int_{-Y}^{+Y} J_v(Y)\,\mathrm{d}Y = \bar{S}^{-1}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi^{-1}\eta^2\zeta^2\left[\frac{\sin\bar{\Omega}(\eta-\xi)}{\eta-\xi}-\frac{\sin\bar{\Omega}\eta}{\eta}\right] \times \hat{A}(t_1-\zeta)\,\hat{A}(t_1-\eta)\,\hat{A}^*(t_1-\zeta-\xi)\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(\eta-\xi)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

In order to calculate the main contributions to the above integrals, as $\bar{\Omega}\to\pm\infty$, we split the kernel into the form

$$\xi^{-1}\eta^2\zeta^2\left[\frac{\sin\bar{\Omega}(\eta-\xi)}{\eta-\xi}-\frac{\sin\bar{\Omega}\eta}{\eta}\right] = \eta\zeta^2\frac{\sin\bar{\Omega}(\eta-\xi)}{\eta-\xi}-\eta\zeta^2\cos\bar{\Omega}\eta\frac{\sin\bar{\Omega}\xi}{\xi} -2\eta\zeta^2\sin\bar{\Omega}\eta\frac{\sin^2\frac{1}{2}\bar{\Omega}\xi}{\xi}. \quad (\mathrm{A}\ 6)$$

The first term represents the contribution from $\xi=\eta$, the second and the third represent the contributions from $\xi=\eta=0$. After the kernel is so split and factorized, the contributions from different stationary points can be identified. To obtain the asymptote of the multiple integral, we now essentially need to deal with the asymptotes of several single integrals. By using a Mellin transform method (Bleistein & Handelman 1990), we find that only the first term contributes an $O(1)$ jump, i.e.

$$\int_{-\infty}^{+\infty} J_v(Y)\,\mathrm{d}Y = j_0\int_0^{+\infty}\int_0^{+\infty} \xi\zeta^2\hat{A}(t_1-\xi)\,\hat{A}(t_1-\zeta)\,\hat{A}^*(t_1-\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta.$$

After further transformations, this can be rewritten

$$\int_{-\infty}^{+\infty} J_v(Y)\,\mathrm{d}Y = 2j_0\int_0^{+\infty}\int_0^{+\infty} [\xi^3+\tfrac{3}{2}\xi^2\eta+\tfrac{1}{2}\xi\eta^2]\,\hat{A}(\tau_1-\xi)\,\hat{A}(t_1-\eta)\,\hat{A}^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \quad (\mathrm{A}\ 7)$$

Appendix B

In (4.75), we have that

$$I_v^{(b)} = \int_0^{+\infty}\int_0^{+\infty} \xi^2 \hat{A}^*(t_1-\xi-\eta)\,\hat{B}(t_1-\eta)\,\mathrm{e}^{\mathrm{i}\Omega(\eta-\xi)}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$I_v = \alpha \bar{S}^{-5}\sin^{-4}\theta\,\hat{U}^*_{1,YY}\,\hat{V}^{(2,0)}_{2,Y},$$

$$I_1 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi\zeta^2\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+v-\zeta)}\,\hat{A}(t_1-v)\,\hat{A}(t_1-v-\xi)\,\hat{A}^*(t_1-v-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}v,$$

$$I_2 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta^2\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+\eta+v-\zeta)}\,\hat{A}(t_1-v-\eta)\,\hat{A}(t_1-v-\xi)\,\hat{A}^*(t_1-v-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v.$$

Substituting the expressions for $\hat{U}^*_{1,YY}$ and $\hat{V}^{(2,0)}_{2,Y}$ into I_v, we have that

$$I_v = -\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta^2[\xi^2+4\sin^2\theta(\xi+\eta)\eta]\,(\xi+2\eta)^{-1}[\mathrm{e}^{-\mathrm{i}\bar{\Omega}(\xi+2\eta-\zeta)}-\mathrm{e}^{\mathrm{i}\bar{\Omega}\zeta}]$$
$$\times \hat{A}(t_1-\eta)\,\hat{A}(t_1-\xi-\eta)\,\hat{A}^*(t_1-\zeta)\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(\xi+2\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

Integration with respect to Y yields

$$\int_{-Y}^{+Y} I_v(Y)\,\mathrm{d}Y$$
$$= -2\hat{S}^{-1}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta^2[\xi^2+4\sin^2\theta(\xi+\eta)\eta)](\xi+2\eta)^{-1}\left[\frac{\sin\bar{\Omega}(\zeta-\xi-2\eta)}{\zeta-\xi-2\eta}-\frac{\sin\bar{\Omega}\zeta}{\zeta}\right]$$
$$\times \hat{A}(t_1-\eta)\,\hat{A}(t_1-\xi-\eta)\,\hat{A}^*(t_1-\zeta)\,\mathrm{e}^{-\mathrm{i}\bar{\omega}(\xi+2\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

Here again the kernel is split up as follows:

$$\zeta^2[\xi^2+4\sin^2\theta(\xi+\eta)\eta](\xi+2\eta)^{-1}\left[\frac{\sin\bar{\Omega}(\zeta-\xi-2\eta)}{\zeta-\xi-2\eta}-\frac{\sin\bar{\Omega}\zeta}{\zeta}\right]$$
$$= \zeta[\xi^2+4\sin^2\theta(\xi+\eta)\eta]\frac{\sin\bar{\Omega}(\zeta-\xi-2\eta)}{\zeta-\xi-2\eta}$$
$$-\zeta[\xi^2+4\sin^2\theta(\xi+\eta)\eta]\cos\bar{\Omega}\zeta\frac{\sin\bar{\Omega}(\xi+2\eta)}{\xi+2\eta}$$
$$-2\zeta[\xi^2+4\sin^2\theta(\xi+\eta)\eta]\sin\bar{\Omega}\zeta\frac{\sin^2\frac{1}{2}\bar{\Omega}(\xi+2\eta)}{\xi+2\eta}. \qquad \text{(B 1)}$$

Following a similar procedure to that used in simplifying J_v, we can obtain

$$\int_{-\infty}^{+\infty} I_v(Y)\,\mathrm{d}Y = -2j_0\int_0^{+\infty}\int_0^{+\infty}[\eta^2+4\sin^2\theta(\xi+\eta)\xi](2\xi+\eta)$$
$$\times \hat{A}(t_1-\xi)\,\hat{A}(t_1-\xi-\eta)\,\hat{A}^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad \text{(B 2)}$$

The jumps related to the integrals $I_v^{(b)}$, I_1 and I_2 are straightforward to calculate, and are

$$\int_{-\infty}^{+\infty} I_v^{(b)}(Y)\,\mathrm{d}Y = 2j_0\int_0^{+\infty} \xi^2\hat{A}^*(\tau_1-2\xi)\,\hat{B}(t_1-\xi)\,\mathrm{d}\xi, \tag{B 3}$$

$$\int_{-\infty}^{+\infty} I_1(Y)\,\mathrm{d}Y = 2j_0\int_0^{+\infty}\int_0^{+\infty} (\xi+\eta)^2\eta\,\hat{A}(t_1-\xi)\,\hat{A}(t_1-\xi-\eta)\,\hat{A}^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{B 4}$$

and

$$\int_{-\infty}^{+\infty} I_2(Y)\,\mathrm{d}Y = 4j_0\int_0^{+\infty}\int_0^{+\infty} [\tfrac{7}{3}\xi^3+3\xi^2\eta+\xi\eta^2] \times\hat{A}(t_1-\xi)\,\hat{A}(t_1-\xi-\eta)\,\hat{A}^*(t_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{B 5}$$

Appendix C

In this appendix, we solve for the function $\tilde{V}_3$ satisfying

$$\hat{\mathrm{L}}_0^{(3)}\tilde{V}_{3,YY} = \tilde{M}_Y^{(1)}+\tilde{M}_Y^{(2)}+\tilde{M}_Y^{(3)}, \tag{C 1}$$

where

$$\tilde{M}_Y^{(1)} = -\tfrac{3}{2}\hat{A}\hat{V}_{2,YYY}^{(2,0)}-\beta\hat{A}\hat{W}_{2,YY}^{(2,2)},\quad \tilde{M}_Y^{(2)} = 8\beta^2(\hat{W}_1\hat{W}_2^{(2,2)})_Y+4\beta\hat{W}_1\hat{V}_{2,YY}^{(2,0)},$$
$$\tilde{M}_Y^{(3)} = 2\beta\hat{W}_{1,Y}\hat{V}_{2,Y}^{(2,0)}-2\beta\hat{W}_{1,YY}\hat{V}_2^{(2,0)}.$$

The solution driven by $\tilde{M}_Y^{(1)}$ is

$$\bar{S}^4\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\tilde{K}_v^{(1)}(\xi,\eta,\zeta)\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\zeta+3\eta)}\hat{A}(t_1-\eta) \times\hat{A}(t_1-\eta-\zeta)\,\hat{A}(t_1-\eta-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{C 2}$$

where the kernel is

$$\tilde{K}_v^{(1)}(\xi,\eta,\zeta) = -\xi(\xi+2\zeta)(2\xi+\zeta)+4\sin^2\theta[-\tfrac{3}{2}\zeta(\xi+\zeta)(\xi+2\zeta)]. \tag{C 3}$$

The solution forced by $\tilde{M}_Y^{(2)}$ is

$$-4\bar{S}^4\sin^4\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\{[2\xi^2+2\xi\eta+\xi\zeta]+4\sin^2\theta(\xi+\eta)\eta\}\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+\zeta+2\eta+3v)} \times\hat{A}(t_1-v-\eta)\,\hat{A}(t_1-v-\zeta)\,\hat{A}(t_1-v-\eta-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v.$$

After some simplification it can be written:

$$-4\bar{S}^4\sin^4\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\tilde{K}_v^{(2)}(\xi,\eta,\zeta)\,\mathrm{e}^{\mathrm{i}\Omega(\xi+2\zeta+3\eta)}\hat{A}(t_1-\eta) \hat{A}(t_1-\eta-\zeta)\,\hat{A}(t_1-\eta-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{C 4}$$

where

$$\tilde{K}_v^{(2)}(\xi,\eta,\zeta) = 3(\xi+\zeta)\eta^2+2(2\xi^2+4\xi\zeta+3\zeta^2)\eta+4\sin^2\theta[\eta^3+(\xi+2\zeta)\eta^2+\zeta(\xi+\zeta)\eta]. \tag{C 5}$$

Before seeking the solution driven by $\tilde{M}_Y^{(3)}$, we write $\tilde{M}_Y^{(3)}$ as follows after using (4.35):

$$\tilde{M}_Y^{(3)} = -2\beta\hat{W}_{1,YY}\hat{B}+[\mathrm{i}\beta\bar{S}^{-1}\hat{W}_{1,YY}\hat{\mathrm{L}}_0^{(2)}\hat{V}_{2,Y}^{(2,0)}+2\beta\hat{W}_{1,Y}\hat{V}_{2,Y}^{(2,0)}]+2\beta\bar{U}_y^{-1}\hat{W}_{1,YY}S_{11}^{(2,0)}.$$

The solution driven by $\tilde{M}_Y^{(3)}$ is then

$$\tilde{V}^{(3)}_{3,YY} = \bar{S}^4 \sin^4\theta[\tilde{I}_v + \tilde{I}_1 + 2\sin^2\theta \tilde{I}_2] + 2\mathrm{i}\bar{S}^3 \sin^2\theta \tilde{I}_v^{(b)},$$

where
$$\tilde{I}_v = \mathrm{i}\beta\bar{S}^{-5}\sin^{-4}\theta \hat{W}_{1,YY}\hat{V}^{(2,0)}_{2,Y},$$

$$\tilde{I}_1 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \xi\zeta^2 \,\mathrm{e}^{-\mathrm{i}\Omega(\xi+3v+\zeta)}\hat{A}(t_1-v)\hat{A}(t_1-v-\xi)\hat{A}(t_1-v-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}v,$$

$$\tilde{I}_2 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta^2 \,\mathrm{e}^{-\mathrm{i}\Omega(\xi+\eta+3v+\zeta)}\hat{A}(t_1-v-\eta)\hat{A}(t_1-v-\xi)\hat{A}(t_1-\nu-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}v,$$

$$\tilde{I}_v^{(b)} = \int_0^{+\infty}\int_0^{+\infty} \xi^2\hat{A}(t_1-\xi-\eta)\hat{B}(t_1-\eta)\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+3\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

The integrals $\tilde{I}_1$ and$\tilde{I}_2$ can be rewritten:

$$\tilde{I}_1 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta(\xi+\zeta)(\xi+2\zeta)\,\mathrm{e}^{\mathrm{i}\Omega(\xi+2\zeta+3\eta)}\hat{A}(t_1-\eta) \times\hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \quad (\mathrm{C}\ 6)$$

$$\tilde{I}_2 = 2\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} [\eta^3+(\xi+2\zeta)\eta^2+(\xi^2+2\xi\zeta+2\zeta^2)\eta]\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\zeta+3\eta)} \times\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \quad (\mathrm{C}\ 7)$$

The solution driven by all three terms can be written as

$$\tilde{V}_{3,YY} = \tilde{V}^{(r)}_{3,YY} + \tilde{V}^{(s)}_{3,YY} + \tilde{V}^{(b)}_{3,YY}, \quad (\mathrm{C}\ 8)$$

where
$$\tilde{V}^{(s)}_{3,YY} = \bar{S}^4\sin^4\theta\tilde{I}_v = \mathrm{i}\beta\hat{S}^{-1}\hat{W}_{1,YY}\hat{V}^{(2,0)}_{2,Y}, \quad (\mathrm{C}\ 9)$$

$$\tilde{V}^{(b)}_{3,YY} = 2\mathrm{i}\bar{S}^3\sin^2\theta\tilde{I}_v^{(b)}, \quad (\mathrm{C}\ 10)$$

$$\tilde{V}^{(r)}_{3,YY} = \bar{S}^4\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \tilde{K}_v(\xi,\eta,\zeta)\,\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\zeta+3\eta)} \times\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \quad (\mathrm{C}\ 11)$$

and

$$\begin{aligned}\tilde{K}_v(\xi,\eta,\zeta) = &-\xi(2\xi+\zeta)(\xi+2\zeta)\\ &+4\sin^2\theta[-\tfrac{5}{4}\zeta(\xi+\zeta)(\xi+2\zeta)-3(\xi+\zeta)\eta^2-(4\xi^2+8\xi\zeta+6\zeta^2)\eta]\\ &+4\sin^4\theta[-3\eta^3-3(\xi+2\zeta)\eta^2+(\xi^2-2\xi\zeta-2\zeta^2)\eta].\end{aligned} \quad (\mathrm{C}\ 12)$$

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